Centrosymmetric Solutions to Chessboard Separation Problems

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Abstract

Chessboard separation problems are modifications to classic chessboard problems, such as the N queens problem, in which obstacles are placed on the chessboard. The $N+k$ queens problem requires placements of k pawns and $N+k$ mutually non-attacking queens on an N-by-N chessboard. Here we examine centrosymmetric (half-turn symmetric) and doubly centrosymmetric (quarter-turn symmetric) solutions to the $N+k$ queens problem. We also consider solutions in which the queens and pawns exhibit different types of symmetry.

Keywords: combinatorial algorithms, N queens problem, symmetry
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1 Introduction

According to standard chess rules, a queen can move any number of squares in a straight line vertically, horizontally, or diagonally, as long as no other piece lies in its path. In 1848, Max Bezzel proposed the 8 queens problem, which calls for a placement of eight queens on an 8×8 chessboard so that no two queens "attack" each other (i.e., no queen lies in another queen's path) [2]. According to [1], in 1869 F. J. E. Lionnet first discussed the N queens problem, which calls for a placement of N nonattacking queens on an $N \times N$ board.

The N queens problem and several variations appear extensively through the mathematics and computer science literatures. In mathematics, the problem has been connected to topics such as graph-theoretic domination, integer programming, and magic squares. In computer science, the problem is an example of a generalized exact cover problems and is used as a model for backtracking programming techniques (including the dancing links method popularized by Knuth in [11]), constraint programming, parallel programming, and neural nets. Collections of references to the N queens problem can be found in the survey article [1] and an online bibliography [12]. We also refer the interested reader to [9] and [18].

In this paper we consider special solutions to the "$N+k$ queens problem", which calls for placing $N+k$ queens 'Q' and k pawns 'P' on an $N \times N$ board so that no two queens attack each other. It was conjectured in [8] and proved in [6] that for each $k \geq 0$, there is a number $N(k)$ depending on k such that if $N > N(k)$ then the $N+k$ queens problem has at least one solution. In [7] algorithms that count the number of solutions to the $N+k$ queens problem for various values of N and k were presented and compared. The first-named author of this paper has considered symmetric solutions to the $N+k$ queens problem in [5] and showed that all solutions to the $N+k$ queens problem (where $N > 1$) are of one of the following three types:

1. *Ordinary* solutions, which are not symmetric under rotation. Figure 1 is an example of an ordinary solution.

2. *Centrosymmetric* solutions, which are symmetric with respect to half-turn rotations but are not symmetric under quarter-turn rotations. Figure 2 is an example of a centrosymmetric solution.

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Q							
					Q		
			Q				
	Q		P			Q	
			Q				
							Q
		Q					
				Q			

Figure 1: Ordinary solution to the 8+1 queens problem

			Q				
						Q	
				Q			
		Q		P			Q
Q			P		Q		
			Q				
	Q						
				Q			

Figure 2: Centrosymmetric solution to the 8+2 queens problem

3. *Doubly centrosymmetric* solutions, which are symmetric with respect to quarter-turn rotations. Figure 3 is an example of a doubly centrosymmetric solution.

	Q			
				Q
		Q		
Q				
			Q	

Figure 3: Doubly centrosymmetric solution to the 5+0 queens problem

Note that Theorem 2 of [5] shows that no solution to the $N + k$ queens problem, except where $N = 1$, is symmetric with respect to horizontal, vertical, or diagonal reflection.

In this paper we further examine symmetric solutions to the $N + k$ queens problem and present and compare algorithms that search only for such solutions. We also consider $N + k$ queens problem solutions for which the queens and pawns exhibit different types of symmetry.

2 Existence results

In this section we present theorems that will narrow the search for centrosymmetric and doubly centrosymmetric solutions to the $N + k$ queens problem.

2.1 Centrosymmetric

First we observe that for a centrosymmetric $N + k$ queens solution, either the board must be of odd order or there must be an even number of pawns.

Proposition 2.1 *For an $N + k$ queens problem, no centrosymmetric solutions exist for N even and k odd.*

Proof. If N is even, the number of pawns in the left half must be equal to that in the right half. Therefore, the number of pawns must be even. ■

Kraitchik [13] observes on page 248 that for a centrosymmetric solution to the N queens problem with N odd, there must be a queen in the central cell. Observe that if N is odd and k is even, then for a centrosymmetric solution of the $N = k$ queens problem the central square also must be occupied by a queen. However, we show on an odd order board with an odd number of pawns that it is a pawn which must occupy the central square. Rows and columns are numbered $0, 1, \ldots, N - 1$ with the upper left square labeled $(0, 0)$.

Proposition 2.2 *For an $N+k$ queens problem with N odd and k odd, centrosymmetric solutions must have a pawn in board square $(\lfloor N/2 \rfloor, \lfloor N/2 \rfloor)$.*

Proof. Since k is odd, at least one pawn (say its location is (i, j) with $0 \leq i, j < N$) must be invariant under a half-turn rotation. But then $(i, j) = (N - 1 - i, N - 1 - j)$ and so $i = j = (N - 1)/2$. Thus we have a pawn in the central square $((N - 1)/2, (N - 1)/2)$. ■

2.2 Doubly Centrosymmetric

Kraitchik [13] notes on pages 248–249 that a doubly centrosymmetric N queens solution can occur only when N is congruent to 0 or 1 modulo 4. We present the analogous observation for $N + k$ queens solutions.

Proposition 2.3 *Let $N \geq 1$ and $k \geq 0$ be integers for which the $N + k$ queens problem has a doubly centrosymmetric solution. Then N and k must satisfy one of the following conditions:*

1. *$N \equiv 0$ (mod 4) and $k \equiv 0$ (mod 4)*
2. *$N \equiv 1$ (mod 4) and $k \equiv 0$ (mod 4)*
3. *$N \equiv 3$ (mod 4) and $k \equiv 1$ (mod 4)*

Proof. Since a doubly centrosymmetric solution is invariant under any number of quarter turn rotations, if such a solution has a piece at square (a, b) (with $0 \leq a, b < N$) then it must also have pieces of the same type at squares $(b, N-1-a)$, $(N-1-a, N-1-b)$, and $(N-1-b, a)$. Unless $(a, b) = (\lfloor N/2 \rfloor, \lfloor N/2 \rfloor)$ and N is odd, the four squares listed above are distinct. So, the number of queens is congruent to either 0 or 1 (mod 4), and the number of pawns is also congruent to either 0 or 1 (mod 4).

The number of pawns in a solution to the $N + k$ queens problem is k, so we must have $k \equiv 0$ (mod 4) or $k \equiv 1$ (mod 4). The number of queens in a solution to the $N + k$ queens problem is $N + k$, so either $N + k \equiv 0$ (mod 4) or $N + k \equiv 1$ (mod 4). If N is even, clearly $N + k \equiv 0$ (mod 4) and $k \equiv 0$ (mod 4), so $N \equiv 0$ (mod 4), and we have condition 1. If N is odd, and the middle square is empty, we have an even number of pawns and an even number of queens, which leads to a contradiction with the parity of N. If N is odd and the middle square has a queen, then $N + k \equiv 1$ (mod 4) and $k \equiv 0$ (mod 4), so $N \equiv 1$ (mod 4) and we have condition 2. If N is odd and the middle square has a pawn, then $k \equiv 1$ (mod 4) and $N + k \equiv 0$ (mod 4), so $N \equiv 3$ (mod 4) and we have condition 3. ■

There is one further special case to consider.

Proposition 2.4 *There are no doubly centrosymmetric solutions to the $N + 1$ queens problem.*

Proof. Suppose we had a doubly centrosymmetric solution to the $N + 1$ queens problem for some N. By the proof of Proposition 2.3, we can conclude N is odd and that the pawn is in the central square at $(\lfloor N/2 \rfloor, \lfloor N/2 \rfloor)$. In order to have $N + 1$ mutually nonattacking queens, the central column must have at least one queen, say at $(a, \lfloor N/2 \rfloor)$, with $0 \leq a < \lfloor N/2 \rfloor$. Since the solution is invariant under a half-turn rotation, there is a queen at $(\lfloor N/2 \rfloor, N - 1 - a)$. But we now have two queens on the same diagonal with no pawn between them, which contradicts our assumption that the queens do not attack each other. ■

2.3 Symmetry of queens different from that of pawns

We next consider $N + k$ queens solutions where the symmetry of the queens is different from the symmetry of the pawns. For example, Figure 4 (left) is a 14+4 queens solution where the queens are arranged centrosymmetrically but the pawns have no symmetry. Figure 4 (right) is a 7+2 queens solution where the queens are doubly centrosymmetric while the pawns are merely centrosymmetric.

Recall that if $k \neq 1$ then there are no $N + k$ queens solutions that are symmetric with respect to a reflection. If we require only the queens to be symmetric, we find that there are solutions where the queens are symmetric with respect to vertical or horizontal reflection but the pawns are not symmetric. The smallest N for which such solutions occur is $N = 21$; Figure 5 is an example.

However, the following proposition shows that in many cases there is no possibility of reflective symmetry of the queens.

Proposition 2.5 *Given a solution to the $N + k$ queens problem (with $N > 1$),*

1. *the queens are not symmetric with respect to reflection across a diagonal,*
2. *if N is even, the queens are not symmetric with respect to vertical or horizontal reflection, and*
3. *if $N = 2s + 1$ and the queens are symmetric with respect to vertical or horizontal reflection, then $k \geq s + 1$.*

Proof.

1. We repeat the argument from Theorem 2 in [5], noting it does not require the pawns to be symmetric. Suppose we have an $N + k$ queens solution where the queens are symmetric with respect to the main diagonal (i.e., from upper left corner to lower right corner). Since the first piece in each row and column

						Q							
				Q									
		Q		P			Q						
												Q	
			Q			P		Q					
													Q
				Q									
									Q				
Q													
					Q				P	Q			
	Q												
						Q	P				Q		
									Q				
							Q						

		Q				
				Q		
	Q			P		Q
			Q			
Q		P			Q	
		Q				
				Q		

Figure 4: Left: queens centrosymmetric, pawns not Right: queens doubly centrosymmetric, pawns centrosymmetric

must be a queen, there is no pawn in the upper left corner square. We prove by induction that the board has no pawns at all: Suppose the upper left $r \times r$ squares have no pawns and that there is a pawn in the $r+1$ column. There must be a queen above that pawn, say at $(t, r+1)$. By the assumed diagonal symmetry there is a queen at $(r+1, t)$, which is in the same rising diagonal as $(t, r+1)$. Since the queens are nonattacking, there must be a pawn between $(r+1, t)$ and $(t, r+1)$, contradicting the assumption that the upper left $r \times r$ corner had no pawns. Since the upper left $r \times r$ corner has no pawns, the upper $(r+1) \times (r+1)$ corner has no pawns. By induction, the board has no pawns. So we have a diagonally symmetric N queens solution, which is clearly impossible. (The main diagonal has at most one queen, and any other queen is, by symmetry, on the same rising diagonal as its symmetric duplicate.)

2. Let $N = 2s$ and suppose we have a solution where the queens are symmetric with respect to reflection across the vertical line between the two central columns. Each column must have at least one queen. By symmetry, the queen in one central column is adjacent to a queen in the other central column, contradicting the given that the queens are mutually nonattacking.

3. Let $N = 2s + 1$ and suppose we have a solution where the queens are symmetric with respect to reflection across the central column. If a square in the central column does not contain a queen then the row containing that square must contain at least two queens and therefore at least one pawn. If the central column had more than $s+1$ queens, then there would be two adjacent queens; if there were $s+1$ queens, the two adjacent columns could not have any queens. Thus, the central column must have at least $N - s = s + 1$ squares with no queen. So, the board must have at least $s+1$ pawns.

■

On the other hand, it is possible for the pawns to have symmetries that the queens do not, even reflective symmetries. For example, in Figure 2 the pawns are symmetric with respect to reflection across the main diagonal, but the queens are not.

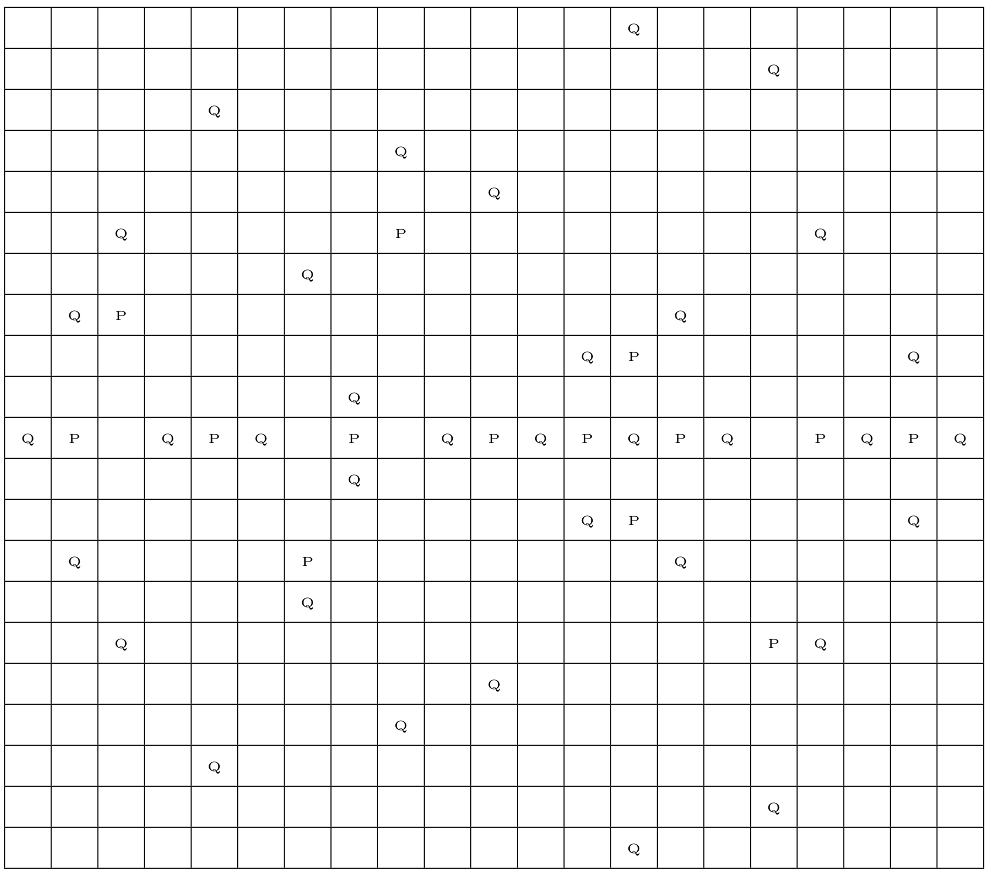

Figure 5: A 21+14 queens solution where the queens are symmetric with respect to reflection across the central row. Note that the pawns are not symmetric.

2.4 Connections to matrices

Recall that a rook is a chess piece that can move any number of spaces in a straight line vertically or horizontally as long as it does not go through another piece. We define the $N + k$ rooks problem to be that of placing $N + k$ nonattacking rooks and k pawns on an $N \times N$ chessboard. It is straightforward to see that $k \leq N(N-1)/2$; however, solutions may only exist if $k \leq \lfloor N/2 \rfloor \cdot (\lceil N/2 \rceil - 1)$. This also provides a (somewhat large) upper bound for the corresponding queens problem.

Each solution to the $N + k$ rooks (or queens) problem can be converted to a matrix representation by replacing each empty square with a 0, each rook (or queen) with a 1, and each pawn with a -1. For example the matrix representation for the solution in Figure 2 is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Every such matrix is an *alternating sign matrix* (ASM) [5], a square matrix consisting of 0s, 1s, and -1s where the sum of the entries in each row and column is 1 and the first and last nonzero entry in each row and column is a 1. Alternating sign matrices have been extensively studied and have applications to statistical mechanics — see [3, 4, 16]. ASMs with various symmetries have also been studied [14]. We consider ASMs in more detail in Section 3.

When we convert a centrosymmetric solution into matrix form we obtain a *centrosymmetric matrix*. Centrosymmetric matrices have applications [15] to pattern recognition, antenna theory, mechanical and electrical systems, and quantum physics.

3 Implementation

Algorithm X is a backtracking algorithm for solving exact cover problems using a data structure that Knuth calls Dancing Links [11]. Previous work [6, 7] found solutions to the general $N + k$ queens problem by enumerating all possible configurations of k pawns on the $N \times N$ chessboard, subject to certain constraints, and for each such configuration using an implementation of Knuth's Algorithm X to place the queens. Following [6] and [7], we refer to this algorithm for solving the $N + k$ queens problem as DLX.

DLX was selected based on performance comparisons to a standard backtracking algorithm using array storage for both the N queens and $N+1$ queens problems. The DLX solver was not reevaluated for the work done in [7], although it was conjectured in [5] that alternating sign matrix solvers might improve performance results. The conjecture observed that the alternating queen and pawn placement is the same problem solved by alternating sign matrices. We developed a backtracking solution (in fact, still inspired by [10]) based on simple arrays that uses the properties of alternating sign matrices. We refer to this algorithm as ASM, and it dramatically outperforms our DLX implementation as k increases.

3.1 Alternating Sign Matrices (ASM)

Whereas the DLX-based algorithm for $N+k$ queens places all of the k pawns before attempting to place any queens, the ASM algorithm (Figure 6) interleaves the placement of queens and pawns, proceeding row-by-row from the top of the chessboard (row 0) to the bottom (row $N-1$). The basic property used is that every pawn must necessarily be between two queens, both horizontally and vertically. The recursive, backtracking function *queensAndPawns* (b, r, c, p) places queens and pawns on the $N \times N$ board b, where there still must be a queen in row r, starting from column c. The function assumes that there are p pawns remaining to be placed. The initial call to the function is *queensAndPawns* $(b, 0, 0, k)$. Figure 7 (left) shows the static search order that is used by ASM on an 8×8 chessboard.

```
queensAndPawns (b, r, c, p)
    if r = N and p = 0 then
        print (b)
        return
    for colQ ← c to N − 1 do
        if noQueenAttacking (b, r, colQ) then
            b[r, colQ] ← Queen
            if p > 0 then
                for colP ← colQ + 1 to N − 2 do
                    if lastPieceInColumn (b, colP) = Queen then
                        b[r, colP] ← Pawn
                        queensAndPawns (b, r, colP + 1, p − 1)
                        b[r, colP] ← Blank
            queensAndPawns (b, r + 1, 0, p)
            b[r, colQ] ← Blank
```

Figure 6: ASM algorithm

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

25	26	27	28	29	30	31	32
17	18	19	20	21	22	23	24
9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1
16	15	14	13	12	11	10	9
24	23	22	21	20	19	18	17
32	31	30	29	28	27	26	25

16	15	12	7	1	8	13	16
13	14	11	6	2	9	14	15
8	9	10	5	3	10	11	12
1	2	3	4	4	5	6	7
7	6	5	4	4	3	2	1
12	11	10	3	5	10	9	8
15	14	9	2	6	11	14	13
16	13	8	1	7	12	15	16

Figure 7: Left: ASM search order on 8×8 board; Middle: adjusted for centrosymmetric solutions; Right: adjusted for doubly centrosymmetric solutions

3.2 ASM versus DLX

The ASM solver has far better performance than DLX as k increases. For $N = 10$, $k = 0$, ASM performs 35,539 recursive calls, while DLX performs only 12,911. This supports our initial finding of DLX outperforming standard array backtracking implementations for N queens. However, using the same value of N but increasing k to 5 results in 885,560 recursive calls for ASM while the number of recursive calls for DLX goes to 32,726,723.

		P							
			P						
		P							

Figure 8: Bad pawn arrangement for first four rows of 10+4 queens problem

The ASM solver is making fewer recursive calls because it has more information during pawn placement than DLX does and can then prune poor pawn placement branches sooner than DLX. For an example illustrating how ASM may prune its search space more effectively, consider the $10 + 4$ queens problem, which has 88 distinct solutions.

The DLX algorithm is an exact cover solver, therefore the pawn configuration must be defined prior to placing queens to 'cover' the resulting chessboard. The DLX algorithm will at some point place its first three pawns in the configuration shown in Figure 8. The given pawn configuration cannot appear in any solution because there is no way to place a queen to the left of each of the indicated pawns without at least one pair of queens having a conflict. For precisely this reason, the ASM algorithm will never consider a partial solution that has this configuration of pawns. However, DLX does not backtrack at this point. Instead, it places the remaining fourth pawn, initially, two spaces to the right of the third pawn. Then it begins a search for a valid placement of queens, which of course fails. When that search completes, DLX then backtracks from the placement of the fourth pawn, placing it now three spaces to the right of the third pawn. The ensuing search for a queen placement also fails. Similar doomed searches are repeated dozens of times because of the configuration of the initial three pawns.

Thus we see that the ASM algorithm affords dramatic search space pruning that is not achieved by the DLX algorithm.

3.3 Adapting ASM for finding centrosymmetric solutions only

One obvious and effective way to modify ASM so that it enumerates only centrosymmetric solutions is as follows: whenever the algorithm instantiates a square s, to instantiate with exactly the same value (`Queen`, `Pawn`, or `Blank`) the square s' that is reached by a half-turn rotation of s. This cuts in half the depth of the resulting backtracking search tree. However, in order to obtain maximum pruning from the rotated instantiations, it is important that are no uninstantiated squares interposed between instantiated squares. In order to ensure that this is the case, we modified the search order on ASM so that it instantiates the middle rows of the board first and works out towards the top and bottom edges as the search proceeds. Figure 7 (middle) shows the search order used on an 8×8 board.

3.4 Adapting ASM for finding doubly centrosymmetric solutions only

In this case, whenever the search algorithm instantiates a square, it instantiates the three additional squares that can be reached through a sequence of quarter-turn rotations. Here again, it is useful to minimize the number of uninstantiated squares that interpose between instantiated squares. Such squares can be avoided altogether by using a search order that instantiates the center square(s) first and spirals outwards. However, such an ordering loses some of the pruning that is afforded to an ordering that proceeds row-by-row (due to enforcement of the alternating sign properties).

Rather than using an ordering that spirals from the center of the board, we found that it is more efficient to again use an ordering that proceeds row-by-row, starting from the middle rows and moving towards the top and bottom edges, although because of the quarter-turn rotations this does leave uninstantiated squares interposing between instantiated squares during the search. In order to deal with this, special consideration is given to configurations where two queens share a diagonal and the only interposing squares (other than `Blank` squares) are uninstantiated squares. In each such case, the algorithm records a constraint that this particular diagonal must eventually receive a pawn. (Note that, because of our search ordering and the symmetries involved in the problem, it is sufficient to record this only for rising diagonals in the upper-left quadrant of the board.) If, at a later point in the search, the remainder of the diagonal is filled in with blanks, then the algorithm backtracks. Furthermore, if at any point in the search it is determined that there are insufficient remaining pawns to meet the remaining diagonal constraints, then the algorithm backtracks. Figure 7 (right) shows the search order used by this algorithm, again on an 8×8 board.

4 Results

Solutions to the $N + k$ queens problem have been obtained using an obviously parallel implementation of Algorithm X as discussed in [7]. Solutions are presented for the centrosymmetric $N + k$ queens problem, doubly centrosymmetric $N+k$ queens problem, and the centrosymmetric queens, non-centrosymmetric pawns $N + k$ queens problem. These results were obtained using an Intel T2300, 1.66GHz computer with 3GB of RAM running Windows XP.

$N \backslash k$	0	1	2	3	4	5	6
7	8	4	4	0	0	0	0
8	4	–	4	–	0	–	0
9	16	20	16	4	0	0	0
10	12	–	8	–	0	–	0
11	48	72	124	32	32	36	4
12	72	–	52	–	20	–	0
13	128	200	568	492	564	260	144
14	420	–	1008	–	804	–	184
15	1240	2608	6284	6164	9000	6188	5252
16	2936	–	12932	–	16796	–	10156
17	8024	17040	64776	75640	161632	129612	173124
18	18104	–	129540	–	300212	–	308712
19	44184	100536	581016	833412	2404616	2368516	4481844
20	144140	–	1475728	–	5277880	–	8988784
21	374960	952392	6535584	10907512	38493012	46514492	108353584
22	1250692	–	17632292	–	90954904	–	235305576
23	3581240	9969216	80020952	151250132	645522628	911776136	
24	11671752	–	225082516	–		–	
25	34129328	101557176	978457836	2070351384			
26	115718268	–	2979154076	–		–	
27	320403024						
28	1250901440	–		–		–	
29	3600075088						

Table 1: Centrosymmetric Solutions, $7 \leq N \leq 29$, $0 \leq k \leq 6$

4.1 Centrosymmetric

The centrosymmetric $N + k$ queens problem identifies solutions to the $N + k$ queens problem for which the queen and pawn locations are invariant under half-turn rotations but not quarter-turn rotations. Table 1 and Table 2 list the number of solutions to the centrosymmetric $N + k$ queens problem. A '–' is used to indicate that no solution can exist for the combination of queens and pawns by one of the results from Section 2. Empty cells refer to timeouts in which computations were halted after 24 hours.

$N \backslash k$	7	8	9	10	11	12	13	14	15	16	17
13	52	8	8	0	0	0	0	0	0	0	0
14	–	16	–	0	–	0	–	0	–	0	–
15	2364	1744	456	352	60	36	8	4	0	0	0
16	–	2948	–	560	–	60	–	8	–	0	–
17	105996	99716	45728	32540	11852	7124	2288	1144	300	102	32
18	–	170624	–	57260	–	11856	–	1664	–	164	–
19	3356828	4471168	2719996	2676960	1355144	1032560	447766				
20	–	8608274	–	5128456	–		–		–		–
21	102089260	170752652	131516964								

Table 2: Centrosymmetric Solutions, $13 \leq N \leq 21$, $7 \leq k \leq 17$

4.2 Doubly Centrosymmetric

The doubly centrosymmetric $N+k$ queens problem identifies solutions to the $N+k$ queens problem for which the queen and pawn locations are invariant under quarter-turn rotations. Apart from the two well-known solutions for the $4+0$ problem, no other solutions exist for $N < 12$. Tables 3 and 4 give the total number of solutions of the doubly centrosymmetric $N+k$ queens problem.

Kraitchik [13] shows on pages 248–250 that the number of doubly centrosymmetric solutions to the N queens problem is divisible by 2^s. With pawns on the board, we obtain the following generalizations.

Proposition 4.1 *Suppose $N = 4s + p$ with $p \in \{0, 1, 3\}$, $k = 4t + r$, $t \geq 1$, $r \in \{0, 1\}$, and $N \geq 7k$. Then the number of doubly centrosymmetric solutions to the $N + k$ queens problem is divisible by 2^{s-7t-r}.*

Proof. Given a doubly centrosymmetric solution, mark all queens that attack a pawn. (For the diagonals, only those that have more than one queen need to be considered.) Every pawn causes at most eight queens to be marked; note that some queens may be marked by several pawns. We conclude that if there is no queen in the central square or if there is no central square then $8k$ is an upper bound for the number of marked queens. If there is a queen in the central square then that queen will also be marked since it is invariant under reflections, in which case there are at most $8k + 1$ marked queens.

If there is no queen in the central square then there are at least $N + k - 8k = N - 7k$ unmarked queens, with at least $(N - 7k)/4$ of these in the first quadrant.

By Proposition 2.3, there is no queen in the central square if $p = r = 0$ or if $p = 3$ and $r = 1$. If $p = r = 0$, then $(N-7k)/4 = (4s-28t)/4 = s-7t$. If $p = 3$ and $r = 1$, then $(N-7k)/4 = (4s+3-28t-7)/4 = s-7t-1$.

If there is a queen in the central square, then $p = 1$ and $r = 0$. There are at least $N+k-8k-1 = N-7k-1$ unmarked queens with at least $(N - 7k - 1)/4 = (4s + 1 - 28t - 1)/4 = s - 7t$ in the first quadrant.

The unmarked queens can all be independently reflected in the main diagonal (the three rotated copies must be moved accordingly; together, these four queens continue to occupy the same rows, columns and diagonals), giving rise to a multiple of 2^{s-7t-r} inequivalent solutions. We conclude that the number of solutions is divisible by 2^{s-7t-r}. ■

4.3 Centrosymmetric Queens, Ordinary Pawns

The number of solutions to the problem of placing queens that are centrosymmetric with ordinary pawns (the constraint that the pawns are neither centrosymmetric nor doubly centrosymmetric) are summarized in Table 5. Given are both the total number of *fundamental* solutions, which cannot be transformed into one another by rotations and/or reflections, and the total number of solutions.

5 Conclusions and Future Work

We presented theoretical and experimental solutions for the centrosymmetric and doubly centrosymmetric $N + k$ queens problem. We also described solutions to problems in which the queens and pawns exhibit different types of symmetry.

Applying alternating sign matrix constraints and restricting queen or pawn placement on the chessboard provides solutions for larger N and k than presented in [7] and this finding provides a framework for future work. We would like to find sufficient conditions for the existence of centrosymmetric solutions as well as study solutions where queens and pawns have different symmetries than presented here. Another interesting problem is to examine symmetric solutions to other chessboard separation problems, such as the domination separation problems discussed in [6] and, given the constraints of doubly symmetric solutions, perhaps extend the problem to three dimensions.

$N\backslash k$	0	4	5	8	9
4	2	0	–	0	–
12	8	4	–	0	–
13	8	0	–	0	–
15	0	–	32	–	2
16	64	130	–	58	–
17	128	232	–	154	–
19	–	–	120	–	258
20	480	1968	–	2058	–
21	704	3148	–	5100	–
23	–	–	1328	–	7328
24	3328	27592	–	69360	–
25	3264	41352	–	141478	–
27	–	–	33088	–	278712
28	32896	520848	–	2561266	–
29	43776	779184	–	4503792	–
31	–	–	624256	–	8133400
32	406784	9608608	–	78866332	–
33	667904	16982560	–	157608036	–
35	–	–	8428160	–	194503760
36	5845504	199182912	–		–
37	8650752	321908928	–		–
39	–	–	119696640	–	
40	77184000		–		–

Table 3: Doubly Centrosymmetric Solutions, $N = 4, 12 - 40$ ($N \not\equiv 2 \pmod 4$), $k = 0, 4, 5, 8, 9$

$N\backslash k$	12	13	16	17	20	21	24	25	28	29	32	33	36
15	–	0	–	0	–	0	–	0	–	0	–	0	–
16	2	–	0	–	0	–	0	–	0	–	0	–	0
17	36	–	10	–	2	–	0	–	0	–	0	–	0
19	–	82	–	6	–	0	–	0	–	0	–	0	–
20	812	–	170	–	14	–	0	–	0	–	0	–	0
21	3258	–	1134	–	264	–	42	–	12	–	2	–	0
23	–	8938	–	4884	–	1346	–	212	–	8	–	0	–
24	82542	–	51988	–	19874	–	4568	–	706	–	70	–	4
25	216412	–	177328	–	87046	–	29344	–		–		–	
27	–	704964	–	850870	–	591342	–		–		–		–
28	5930414	–		–		–		–		–		–	

Table 4: Doubly Centrosymmetric Solutions, $15 \leq N \leq 28$ ($N \not\equiv 2 \pmod 4$), $12 \leq k \leq 36$ ($k \equiv 0, 1 \pmod 4$)

$N\backslash k$	1	2	3	4	5
12	0	0	0	0	0
13	0	0	0	0	0
14	0	16/2	0	16/2	16/2
15	0	0	104/13	248/31	
16	0	0	0	448/56	
17	0	0	0	448/56	
18	0	0	0	8576/1072	
19	0	0	3592/449		
20	0	0	0		
21	0	0	0		

Table 5: Centrosymmetric Queens, Ordinary Pawns, $12 \le N \le 21$, $1 \le k \le 5$

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