Centrosymmetric Solutions to the $N + k$ Queens Problem

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N Queens Problem

- n queens on $n \times n$ chessboard
- no two queens are on same row, column, or diagonal

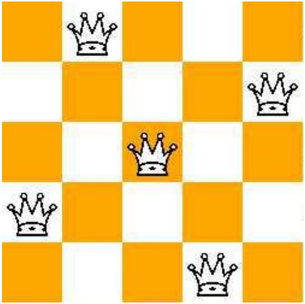


Figure: Solution to 5 queens problem

$N + k$ Queens Problem

- $n + k$ queens, k pawns on $n \times n$ chessboard
- pawn between queens in same row, column, or diagonal

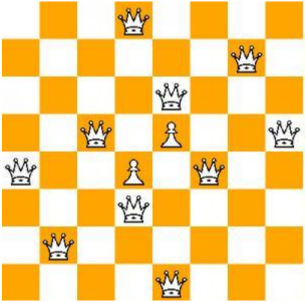
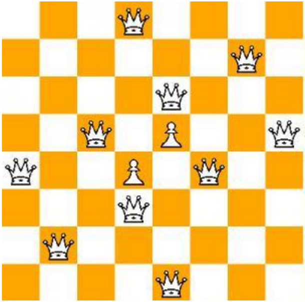


Figure: Solution to 8+2 queens problem

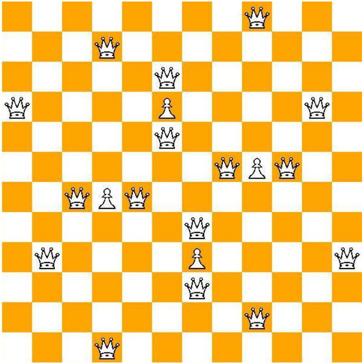
Symmetries

A solution to an $n + k$ queens problem can be *centrosymmetric* (symmetric wrt 180-degree rotations, but not 90-degree rotations)



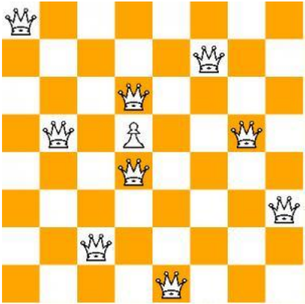
Symmetries

A solution to an $n + k$ queens problem can be *doubly centrosymmetric* (symmetric wrt 90-degree rotations)



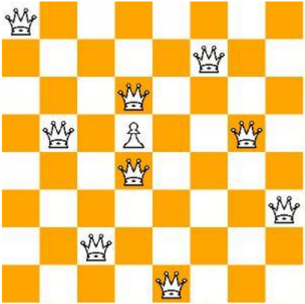
Symmetries

A solution to an $n + k$ queens problem can be *ordinary* (neither centrosymmetric nor doubly centrosymmetric)



Symmetries

A solution to an $n + k$ queens problem can be *ordinary* (neither centrosymmetric nor doubly centrosymmetric)



No solution (with $n > 1$) is symmetric with respect to reflection.

Centrosymmetric solutions

- No such solutions if n even and k odd

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Centrosymmetric solutions

- No such solutions if n even and k odd
- If n odd and k even, queen in central square
- If n odd and k odd, pawn in central square
- Even number of solutions

Doubly centrosymmetric solutions

- Either
  - $n \equiv 0 \pmod 4$ and $k \equiv 0 \pmod 4$,
  - $n \equiv 1 \pmod 4$ and $k \equiv 0 \pmod 4$, or
  - $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$

Doubly centrosymmetric solutions

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  - $n \equiv 0 \pmod 4$ and $k \equiv 0 \pmod 4$,
  - $n \equiv 1 \pmod 4$ and $k \equiv 0 \pmod 4$, or
  - $n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$
- No doubly centrosymmetric solution to $n + 1$ queens problem

Powers of 2 in solution numbers

Proposition

Suppose $n = 4s$ with $s \geqslant 2$, $k = 4t$, and $n \geqslant 7k$. Then the number of doubly centrosymmetric solutions to the $n + k$ queens problem is divisible by 2^{s-7t}.

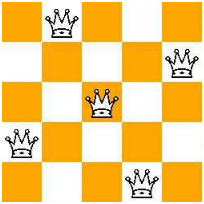

Figure: Note outer ring of squares

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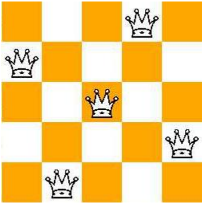

Figure: Note outer ring of squares

Powers of 2 in solution numbers

Proposition

Suppose $n = 4s + 1$ *with* $s \geqslant 4$, $k = 4t$, *and* $n \geqslant 7k$. *Then the number of doubly centrosymmetric solutions to the* $n + k$ *queens problem is divisible by* 2^{s-7t}.

Powers of 2 in solution numbers

Proposition

Suppose $n = 4s + 3$ *with* $s \geqslant 3$, $k = 4t + 1$, *and* $n \geqslant 7k$. *Then the number of doubly centrosymmetric solutions to the* $n + k$ *queens problem is divisible by* 2^{s-7t-1}.

Different symmetries for diff. pieces

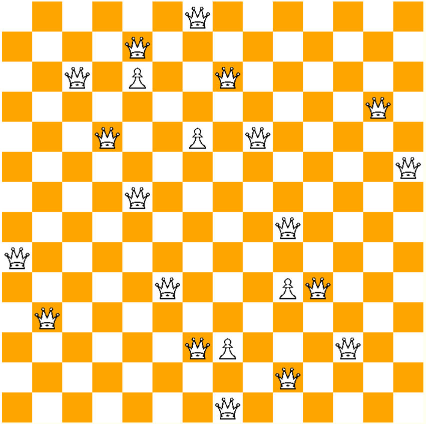

Figure: 14+4 queens solution, queens centrosymmetric, pawns ordinary

Different symmetries for diff. pieces

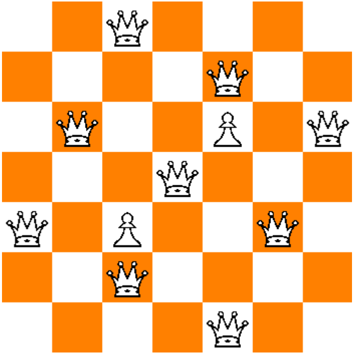

Figure: 7+2 queens solution, queens doubly centrosymmetric, pawns centrosymmetric

Queens with reflective symmetry

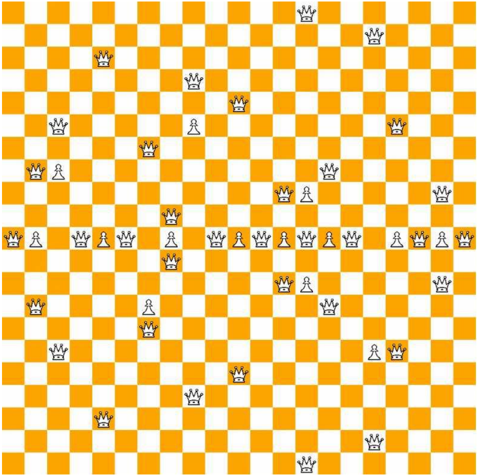

Figure: 21+14 queens solution, queens symmetric wrt reflection across central row

Reflective symmetry for queens

Proposition

Given a solution to the $n+k$ *queens problem (with* $n>1$*),*

1. *the queens are not symmetric with respect to reflection across a diagonal,*

Reflective symmetry for queens

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Given a solution to the $n + k$ queens problem (with $n > 1$),

1. *the queens are not symmetric with respect to reflection across a diagonal,*
2. *if n is even, the queens are not symmetric with respect to vertical or horizontal reflection, and*

Reflective symmetry for queens

Proposition

Given a solution to the $n + k$ queens problem (with $n > 1$),

1. *the queens are not symmetric with respect to reflection across a diagonal,*
2. *if n is even, the queens are not symmetric with respect to vertical or horizontal reflection, and*
3. *if $n = 2s + 1$ and the queens are symmetric with respect to vertical or horizontal reflection, then $k \geqslant s + 1$.*

References

- N+k Queens Problem Page: npluskqueens.info
- n-Queens bibliography: www.liacs.nl/home/kosters/nqueens/
- M. Kraitchik, Mathematical Recreations, 2nd ed., Dover Publications Inc., New York, 1953.