Reflections on the $N + k$ Queens Problem

R. Douglas Chatham

R. Douglas Chatham (d.chatham@moreheadstate.edu) is an associate professor of mathematics at Morehead State University. He received his B.S. from Wake Forest University and his Ph.D. from the University of Tennessee, Knoxville. He currently serves as Newsletter Editor and Webmaster for the Kentucky Section of the MAA.

Given a regular chessboard, can you place eight queens on it, so that no two queens attack each other? More generally, given a square chessboard with N rows and N columns, can you place N queens on it, so that no two queens attack each other?

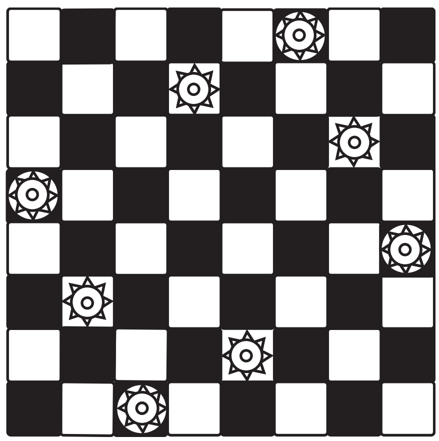


Figure 1. A solution to the eight queens problem.

This puzzle, known as the N queens problem, is old, and famous, and has an extensive history. Here we present a recently formulated elaboration, which we call the $N + k$ queens problem. We describe some of what is known about the $N + k$ queens problem, prove a few new results, and propose some open questions.

The problem

Clearly, we can put no more than N mutually nonattacking queens on the $N \times N$ board, since a row (or column) with two queens has queens that attack each other. However, if other pieces (let's call them pawns) block the attacks, we can place more nonattacking queens. If we place k pawns on the board, we can sometimes place $N + k$ nonattacking queens, but never more than that. So, we have what we call the $N + k$ queens problem:

The $N + k$ Queens Problem. Let $N \geq 1$ and $k \geq 0$ be integers. On an $N \times N$ chessboard, can you place $N + k$ queens and k pawns so that any two queens on the same row, column, or diagonal have at least one pawn between them?

For what N and k are there solutions to the $N + k$ queens problem? If $k = 0$, then we have the N queens problem for which solutions exist for $N = 1$ and $N \geq 4$. Next is $k = 1$. By [**6**, Theorem 1], the $N + 1$ queens problem has solutions for all $N \geq 6$. The proof involves taking known solutions to the M queens problem for some $M < N$ and adding some rows, some columns, some queens, and a pawn in systematic ways to obtain patterns that are verifiably solutions to the $N + 1$ queens problem. Here is one such pattern, which works for even numbers $N \geq 6$ not of the form $6m + 4$:

Pattern. Let $M = N - 2$ and number the rows of the $N \times N$ board $-1, 0, 1, \ldots, M$ and the columns $-2, -1, 0, \ldots, M - 1$. (See Figure 2.) Place the pawn in the square in row $M/2 - 1$ and column -1, i.e., at position $(M/2 - 1, -1)$, and the queens at $(M/2 - 1, -2)$, $(M, -1)$, $(-1, -1)$, $(2i + 1, i)$ for $0 \leq i < M/2$, and $(2i - M, i)$ for $M/2 \leq i < M$.

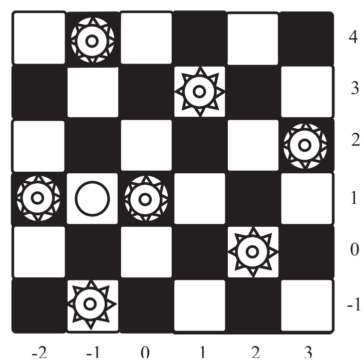


Figure 2. A solution to the 6 + 1 queens problem.

How do we verify such a pattern? We easily check that when two queens are in the same row or the same column, that the pawn is between them. If two queens were in the same rising diagonal, then *row − column* would be the same for both queens. If two queens were in the same falling diagonal, then *row + column* would be the same for both queens. With many "elementary but tedious calculations" [**6**], we verify that each queen has a different *row + column* number and that each queen has a distinct *row − column* number. So for this pattern no two queens are on the same diagonal. Other patterns have similar verifications.

For other k, according to [**4**, Theorem 11], if $N > \max\{87 + k, 25k\}$, then there is a solution to the $N + k$ queens problem. The proof also involves taking known solutions to the M queens problem for some $M < N$, and adding extra rows, columns, queens, and pawns in systematic ways to obtain patterns that can be verified as being $N + k$ queens problem solutions.

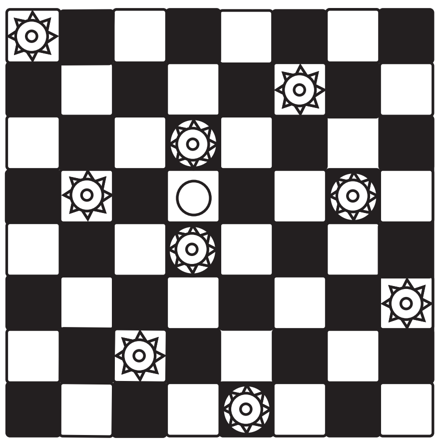
The above result, combined with computer searches, tells us that the $N + 2$ queens problem has solutions for all $N \geq 7$, and that the $N + 3$ queens problem has solutions for all $N \geq 8$ [**4**, p. 8]. This evidence, and the results in Table 1, leads us to believe that the bounds given by [**4**, Theorem 11] can be greatly decreased. In any event, the following question is open.

Open Question 1. For $k > 3$, determine the values of N for which the $N + k$ queens problem has a solution.

Integre Technical Publishing Co., Inc. College Mathematics Journal 40:3 March 12, 2009 2:02 p.m. chatham.tex page 206

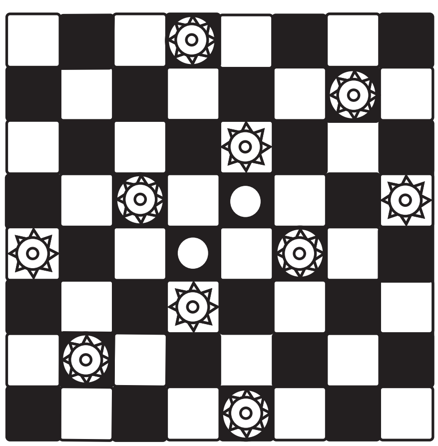
Connections with alternating sign matrices

As demonstrated in Figures 3 and 4, we can transform an $N + k$ queens problem solution into a matrix by replacing each empty square with a 0, each queen with a 1, and each pawn with a -1. Consider an $N + k$ queens problem solution. Each pawn divides its row (or column) in at most two segments, so k pawns create at most k extra row segments. Since there are $N + k$ queens, and only one can go in each segment, so there must be $N + k$ segments and each must have exactly one queen in it. In each row (or column), between any two queens there is a pawn, between any two pawns there is a queen, and the first and last pieces must be queens. So, if you replace each empty square with a 0, each queen with a 1, and each pawn with a -1, you have what is called an *alternating sign matrix (ASM)*: a square matrix consisting of 0s, 1s, and -1s where the sum of the entries in each row and column is 1 and the nonzero entries in each row and column alternate in sign [**1**, p. 3]. However, not all ASMs are $N + k$ queens problem solutions—the ASM definition doesn't require alternation in the diagonals.



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3. A solution to the $8 + 1$ queens problem and its matrix representation.



$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 4. A solution to the $8 + 2$ queens problem and its matrix representation.

If we consider rooks (which move vertically and horizontally, but not diagonally) rather than queens, then we can prove that an ASM represents an $N + k$ *rooks* problem solution and *vice versa*.

Theorem 1. *Let $N \geq 1$ and $k \geq 0$ be integers. There is a one-to-one correspondence between the set of $N \times N$ alternating sign matrices with k entries equal to -1 and the set of solutions to the $N + k$ rooks problem.*

Proof. The argument of the preceding paragraphs implies that every $N + k$ rooks or queens problem solution corresponds to an ASM. Suppose, conversely, that we have an $N \times N$ ASM with k -1s. Since the nonzero entries in each row add up to 1, and the nonzero entries in a row alternate in sign, each row has one more rook than it does pawns, so there are a total of $N + k$ rooks. Since the nonzero entries in each row and column alternate in sign, whenever two rooks are in the same row or column, there is at least one pawn between them. Hence, an ASM represents an $N + k$ rooks problem solution. Thus we have a one-to-one correspondence between the set of $N + k$ rooks problem solutions for a given N and k and the set of $N \times N$ alternating sign matrices with k -1s. ■

Theorem 1, which is new, implies that results about alternating sign matrices are also results about $N + k$ rooks problem solutions. For example, consider the famous Alternating Sign Matrix Conjecture, made by Mills, Robbins, and Rumsey in 1983 [**10**], which declares that the number A_n of alternating sign matrices of order n is given by the formula:

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Zeilberger proved this conjecture with the aid of a computer in 1995 [**17**] and Kuperberg produced a different proof using tools of statistical mechanics in 1996 [**8**]. Now, if we fix N and add up the numbers of $N + k$ rooks problem solutions for all k, the total is A_n. Theorem 1 suggests the following tantalizing question.

Open Question 2. There are boards for which more than k pawns are needed in order to accommodate k extra queens. For example, in her thesis [**18**], Kaiyan Zhao showed that three pawns are needed in order to place six mutually nonattacking queens on a 5×5 chessboard. The matrix for such an arrangement is not an alternating sign matrix. (Two pawns in a row or column need not have a queen between them.) Is there an interesting theory of these "weakly alternating sign matrices"? In particular, is there a nice formula for the number of such matrices of order n?

Symmetry of solutions of the $N + k$ queens problem

There are eight symmetries of the square: rotations of 0, 90, 180, and 270 degrees, vertical reflections, horizontal reflections, and diagonal reflections. The numbers of ASMs with each of these symmetries are mostly known (see [**9**], [**11**], [**13**], and [**14**]), with the exception of diagonal reflection. For any $N \geq 4$, the set of N queens problem solutions can be divided [**16**, pp. 174–175] into three classes:

- *ordinary* solutions, which are symmetric only with respect to 0 degree rotation.
- *centrosymmetric* solutions, which are symmetric with respect to 180 degree rotation, but not 90 degree rotation. Figure 1 is an example of a centrosymmetric solution.
- *doubly centrosymmetric* solutions, which are symmetric with respect to all rotations. Figure 5 is an example.

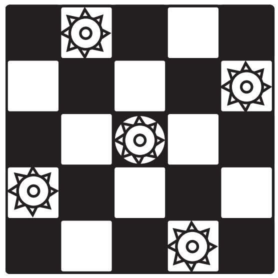


Figure 5. A doubly centrosymmetric solution to the five queens problem.

Note that, except for the trivial solution when $N = 1$, no N queens problem solution is symmetric with respect to a reflection. (Try it, and you'll see that at least one queen will be attacked, usually by its symmetric duplicate.) We now extend this result to the $N + k$ queens problem.

Theorem 2. *No solution to the $N + k$ queens problem (where $N > 1$) is symmetric with respect to a horizontal, vertical, or diagonal reflection.*

Proof. First we deal with vertical mirror symmetry. Suppose we have an arrangement of $N + k$ mutually nonattacking queens and k pawns on an N-by-N board symmetric with respect to a vertical mirror. If N is even, then the queens in the two central columns must be adjacent and thus attacking. If N is odd, then each cell of the central column must be occupied, since otherwise the pieces in the row which are closest to the central empty square are identical, which contradicts the fact that the arrangement is a disguised alternating sign matrix. The central column must then have a queen in its first, third, and every other subsequent row—just as in Figure 6. But then all the squares in the columns next to the central column are attacked by queens. Since each column must have at least one queen, we have queens attacked by other queens, which contradicts the given that none of the queens attack other queens. So we can't have symmetry with respect to a vertical mirror. Repeating the previous argument with our head turned, we also eliminate horizontal mirror symmetry.

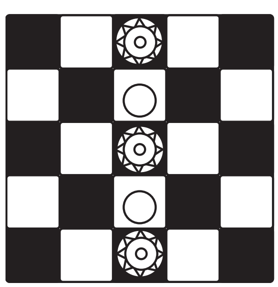


Figure 6. If an $N + k$ queens problem solution were symmetric with respect to a vertical mirror, the central column would look like this.

Next we eliminate diagonal symmetry. Suppose we have an $N + k$ queens problem solution symmetric with respect to the main diagonal. By the properties of an alternating sign matrix, we know that there is no pawn in the cell in the first row and first column. Suppose there is no pawn in the upper left k-by-k corner. (In Figure 7, $k = 3$.) If there is a pawn in the upper left $(k + 1)$-by-$(k + 1)$ corner, say at $(m, k + 1)$, then we have a queen at some $(p, k + 1)$ for $p < m$. (In Figure 7, $m = 2$ and $p = 1$.) By symmetry, we have a queen at $(k + 1, p)$ which is on the same rising diagonal as $(p, k + 1)$. Either those queens attack or there is a pawn between them. If there is a pawn between them, that pawn must be in the upper left k-by-k square. So by induction, there are no pawns on the board at all. We have a diagonally symmetric N queens problem solution, which is already known to be impossible. Again, by turning our heads just right we can eliminate the other kind of diagonal symmetry. ■

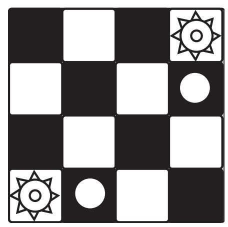


Figure 7. Proof that a diagonally symmetric $N + k$ queens problem solution has no pawns.

Open Question 3. As Table 1 [**3**] indicates, centrosymmetric solutions to the $N + k$ queens problem seem to be rare. Doubly centrosymmetric solutions are even rarer. We've only found 4 doubly centrosymmetric $N + k$ queens problem solutions with

Table 1. In each cell, the first number is the number of centrosymmetric $N + k$ queens problem solutions and the second number is the total number of $N + k$ queens problem solutions.

N	$k = 1$	$k = 2$	$k = 3$	$k = 4$
6	(0, 16)	(0, 0)	(0, 0)	(0, 0)
7	(4, 20)	(4, 4)	(0, 0)	(0, 0)
8	(0, 128)	(4, 44)	(0, 8)	(0, 0)
9	(20, 396)	(16, 280)	(4, 44)	(0, 8)
10	(0, 2288)	(8, 1304)	(0, 528)	(0, 88)
11	(72, 11152)	(124, 12452)	(32, 5976)	(32, 1688)
12	(0, 65172)	(52, 105012)	(0, 77896)	(20, 30936)
13	(200, 437848)	(568, 977664)	(492, 1052884)	(564, 627916)
14	(0, 3118664)	(1008, 9239816)	(0, 13666360)	(804, 11546884)
15	(2608, 23387448)	(6284, 90776620)	(6164, 179787988)	
16	(0, 183463680)	(12932, 897446092)		
17	(17040, 1474699536)			

$k > 0$. (Those solutions occur at $N = 12$ and $k = 4$.) So, how rare are the centrosymmetric and doubly centrosymmetric solutions?

Conclusion

For more information about the N queens problem, see [**7**], [**15**], and [**16**]. For more about the $N + k$ queens problem and related problems, see [**6**], [**4**], and [**5**] (Preprints are available online at [**3**]). For those interested in alternating sign matrices, we suggest [**1**], [**2**], and [**12**].

We conclude with a final open question:

Open Question 4. If we consider a chessboard that's been made a cylinder or a torus by gluing together the appropriate sides, what sort of queens problem solutions and matrices will we get? Also, what sorts of matrices do we get if we look at other pieces, like the king and bishop?

Acknowledgment. The author wishes to thank the anonymous referees and the editor, whose helpful comments vastly improved this paper.

References

1. D. M. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Mathematical Association of America, Washington DC, 1999.
2. D. M. Bressoud and J. Propp, How the alternating sign matrix conjecture was solved, *Notices Amer. Math. Soc.* **46** (1999) 637–646.
3. R. D. Chatham, The $N + k$ Queens Problem Page (2008); available at `http://npluskqueens.info`.
4. R. D. Chatham, M. Doyle, G. H. Fricke, J. Reitmann, R. D. Skaggs, and M. Wolff, Independence and domination separation on chessboard graphs, *J. Combin. Math. Combin. Comput.* **68** (2009) 3–17.
5. R. D. Chatham, M. Doyle, J. J. Miller, A. M. Rogers, R. D. Skaggs, and J. A. Ward, Algorithm performance for chessboard separation problems, *J. Combin. Math. Combin. Comput.* **70** (2009) (to appear).
6. R. D. Chatham, G. H. Fricke, and R. D. Skaggs, The queens separation problem, *Util. Math.* **69** (2006) 129–141.
7. C. Erbas, S. Sarkeshik, and M. M. Tanik, Different perspectives of the N-queens problem, in *Proceedings of the 1992 ACM Annual Conference on Communciations*, CSC '92, Association for Computing Machinery, New York, 1992, 99–108.
8. G. Kuperberg, Another proof of the alternating-sign matrix conjecture, *Internat. Math. Res. Not.* **1996** (1996) 139–150.
9. ———, Symmetry classes of alternating sign matrices under one roof, *Ann. of Math.* (2) **156** (2002) 835–866.
10. W. H. Mills, D. P. Robbins, and H. Rumsey, Jr., Alternating sign matrices and descending plane partitions, *J. Combin. Theory Ser. A* **34** (1983) 340–359.
11. S. Okada, Enumeration of symmetry classes of alternating sign matrices and characters of classical groups, *J. Algebraic Combin.* **23** (2006) 43–69.
12. J. Propp, The many faces of alternating-sign matrices, in *Discrete models: combinatorics, computation, and geometry*, Discrete Math. Theor. Comput. Sci. Proc., AA, Maison Inform. Math. Discrèt. (MIMD), Paris, 2001, 043–058.
13. A. V. Razumov and Y. G. Stroganov, Enumeration of odd-order alternating-sign quarter-turn symmetric matrices, *Teoret. Mat. Fiz.* **149** (2006) 395–408; English translation in *Theoret. and Math. Phys.* **149** (2006) 1639–1650.
14. ———, Enumeration of odd-order alternating-sign half-turn-symmetric matrices, *Teoret. Mat. Fiz.* **148** (2006) 357–386; English translation in *Theoret. and Math. Phys.* **148** (2006) 1174–1198.
15. I. Rivin, I. Vardi, and P. Zimmermann, The n-queens problem, *Amer. Math. Monthly* **101** (1994) 629–639.
16. J. J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, 2004.
17. D. Zeilberger, Proof of the alternating-sign matrix conjecture, *Electron. J. of Combin.* **3** (1996), Research Paper 13.
18. K. Zhao, *The Combinatorics of Chessboards*, Ph.D. dissertation, City University of New York, 1998.

