How Many Mutually Non-attacking, Non-knight Pieces Can We Place on an M-by-N Chessboard?

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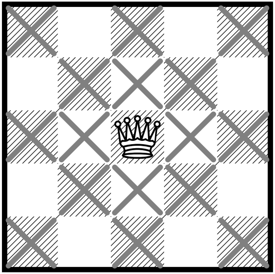
Outline

Section 1

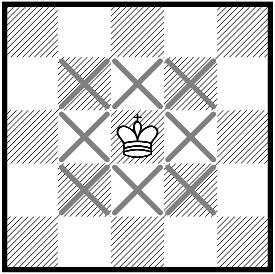
Introduction

Chess piece attacks



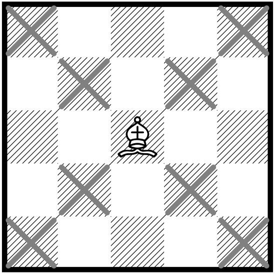
Queen attacks row, column, and diagonals.

Chess piece attacks



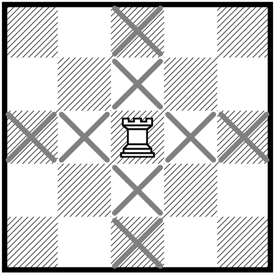
King attacks neighboring squares.

Chess piece attacks



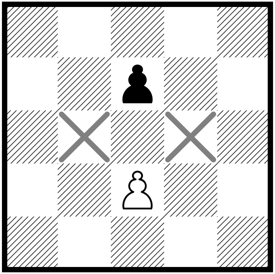
Bishop attacks diagonals.

Chess piece attacks



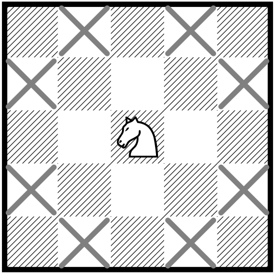
Rook attacks row and column.

Chess piece attacks



White pawns attack diagonally forward.
Black pawns attack diagonally backward.

Chess piece attacks



Knights attack the opposite corner of 2×3 or 3×2 array.

Knight-free capacity

Definition

The **knight-free capacity (kfc)** *of an $m \times n$ chessboard is the maximum number of non-knight pieces we can put on the board such that no piece attacks another.*

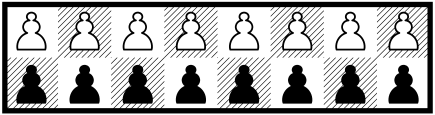
The kfc of a $1 \times p$ or $p \times 1$ board is p.

Knight-free capacity

Definition

The **knight-free capacity (kfc)** *of an $m \times n$ chessboard is the maximum number of non-knight pieces we can put on the board such that no piece attacks another.*

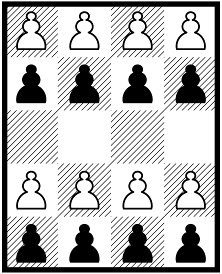
The kfc of a $2 \times p$ board is $2p$.



Section 2

Patterns

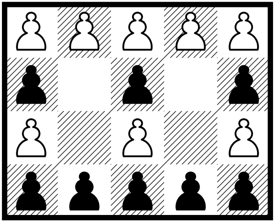
m-layer pattern



Proposition

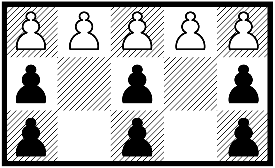
The kfc of an $m \times n$ board is at least $T = T(m,n) = \lceil \frac{2m}{3} \rceil n$.

Loopy pattern



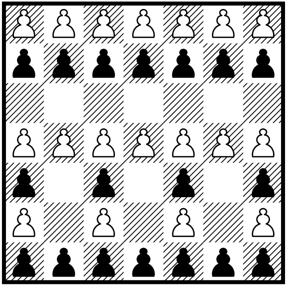
- On $4 \times n$ board with n odd, occupies $3n + 1 = T + 1$ squares.
- On $6 \times n$ board with n odd, occupies $4n + 2 = T + 2$ squares.

Top half loopy



- On $3 \times n$ board with n odd, occupies $2n + 1 = T + 1$ squares.

Loopy pattern covered by k roofs



- On $(3k+4) \times n$ board, where n is odd, occupies $(2k+3)n + 1 = T + 1$ squares.
- On $(3k+6) \times n$ board, where n is odd, occupies $(2k+4)n + 2 = T + 2$ squares.

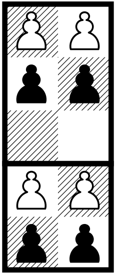
Section 3

Knight-free capacities

Two columns

Proposition

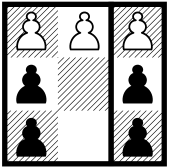
The kfc of an $m \times 2$ board is $\lceil \frac{2m}{3} \rceil 2$.



Three columns

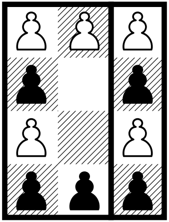
Proposition

The kfc of an $m \times 3$ board is 7 for $m = 3$ and $2m + 2$ for $m > 3$.



3×3 base case

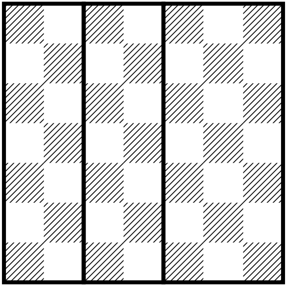
Three columns, continued



1. Top and bottom rows full, or done by induction.
2. Other rows have at least two pieces, or done by induction.
3. Center column empty, except first and last rows.

More columns

- Each 2-column strip holds at most $\lceil \frac{2m}{3} \rceil 2$ pieces.
- The 3-column strip holds at most $2m + 2 = \lceil \frac{2m}{3} \rceil 3 + (2 - i)$ pieces, where $i \equiv m \pmod 3$ and $i = 0, 1$ or 2.



Knight-free capacities for $m \times n$ boards

$m \backslash n$	$2b$	$2b+1$
3	T m-layer	T+1 Top half loopy
$3a$ $a \geq 2$	T m-layer	T+2 Loopy with $a-2$ roofs
$3a+1$	T m-layer	T+1 Loopy with $a-1$ roofs
$3a+2$	T m-layer	T m-layer

Table 1: Knight-free capacity of $m \times n$ board, where $m > 2$, $n > 1$, and $T = \lceil \frac{2m}{3} \rceil n$

Section 4

Counting solutions

How many solutions?

- For a given $m \times n$ board with kfc c, how many arrangements of c non-knight pieces are there?

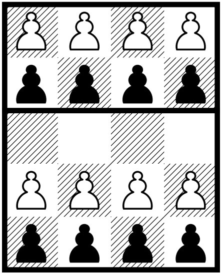
Proposition

For the $1 \times p$ *or* $p \times 1$ *board, there are* 3^p *arrangements.*

How many solutions?

Proposition

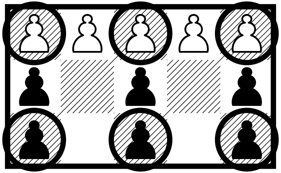
For the $2 \times n$ or $(3a+2) \times 2b$ board, there is 1 arrangement: the m-layer pattern.



How many solutions?

Proposition

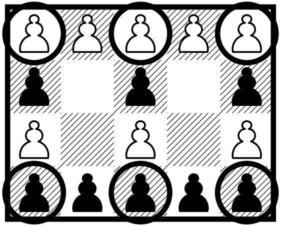
For the $3 \times (2b+1)$ *board, there are* $2(4^{b+1})$ *arrangements: all half-loopy variants.*



How many solutions?

Proposition

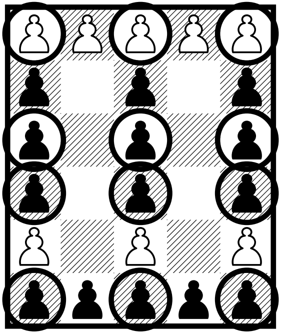
For the $4 \times (2b+1)$ *board, there are* 4^{b+1} *arrangements: all 4-row loopy variants.*



How many solutions?

Proposition

For the $6 \times (2b+1)$ board, there are 16^{b+1} arrangements: all 6-row loopy variants.



How many solutions? (cont.)

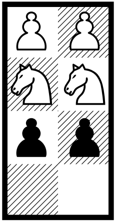
$m \backslash n$	2	3	4	5	6	7	8
2	1	1	1	1	1	1	1
3	121	32	900	128	12100	512	184900
4	36	16	144	64	1024	256	6400
5	1	65	1	1	1	1	1
6	900	256	14161	4096	546121	65536	18688329
7	81	1056	400	128	3136	512	20736
8	1	4225	1	1	1	1	1

Table 2: Numbers of ways to fill the knight-free capacity of $m \times n$ boards

Section 5

Full capacities

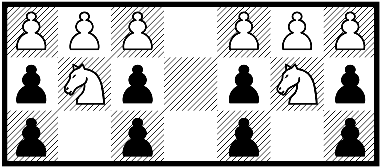
What if we include knights?



Proposition

The "full capacity" of an $m \times 2$ *board is* $2\lceil \frac{3m}{4} \rceil$.

Horse stable pattern



On $3 \times n$ board, occupies

- $2n$ squares if $n \equiv 0 \pmod 4$,
- $2n + 1$ squares if $n \equiv 1$ or $2 \pmod 4$, and
- $2n + 2$ squares if $n \equiv 3 \pmod 4$.

Conjectured full capacities

$m \backslash n$	$4b$	$4b+1$	$4b+2$	$4b+3$
3	T Stable	T+1 Stable	T+1 Stable	T+2 Stable
$3a$ $a \geq 2$	T Stable with $a-1$ roofs	T+2 Loopy with $a-2$ roofs	T+1 Stable with $a-1$ roofs	T+2 Stable with $a-1$ roofs
$3a+1$	T m-layer	T+1 Loopy with $a-1$ roofs	T m-layer	T+1 Loopy with $a-1$ roofs
$3a+2$	T m-layer	T m-layer	T m-layer	T m-layer

Table 3: Conjectured full capacity of $m \times n$ board for $m \geq 3$, $n \geq 4$, where $T = \lceil \frac{2m}{3} \rceil n$

Some References

- D. Chatham, Arrangements of mutually non-attacking chess pieces of mixed type, *Recreational Mathematics Magazine*, **11**(18)(2024), 1 – 16. https://doi.org/10.2478/rmm-2024-0001
- D. Chatham, How many mutually non-attacking, non-knight pieces can we place on an $m \times n$ chessboard?, submitted to *Recreational Mathematics Magazine*.
- MiniZinc website. http://www.minizinc.org

Any questions?

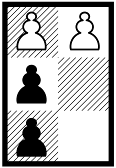
Section 7

Appendix

Essentially different solutions?

Operations that don't make solutions essentially different:

- Flipping across vertical mirror



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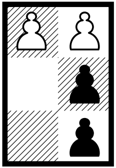
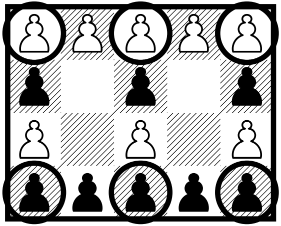


Essentially different solutions?

Operations that don't make solutions essentially different:

- Flipping across vertical mirror
- Changing the color of a pawn with no diagonal neighbors



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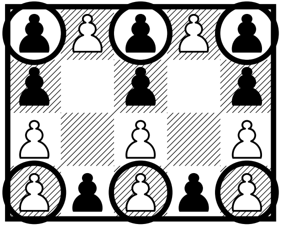
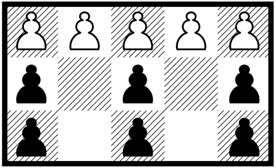


Essentially different solutions?

Operations that don't make solutions essentially different:

- Flipping across vertical mirror
- Changing the color of a pawn with no diagonal neighbors
- Flipping across horizontal mirror and then changing the color of all pawns



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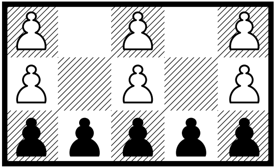

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For $m \times n$ board with knight-free capacity c, how many essentially different attack-free arrangements of c non-knight pieces are there?

Graph-theoretic generalization

Let $G = (V, E)$ be a digraph. Suppose E is covered by a sequence of subsets of E, call it $\mathcal{E} = \{E_i\}_{i=1}^{n}$.

- Let $S \subseteq V$ be a disjoint union of a sequence of subsets $\{S_i\}_{i=1}^{n}$ of V. This set S is $\mathcal{E}$-**cover-independent** or $\mathcal{E}$-**attack-free** if for each $1 \leq i \leq n$ and each $x \in S_i$, for all $y \in S$ we have $(x, y) \notin E_i$.
- The $\mathcal{E}$-**cover-independence number** or $\mathcal{E}$-**capacity** of G is the maximum cardinality of an $\mathcal{E}$-attack-free subset of V.