The Maximum Queens Problem on a Rectangular Board

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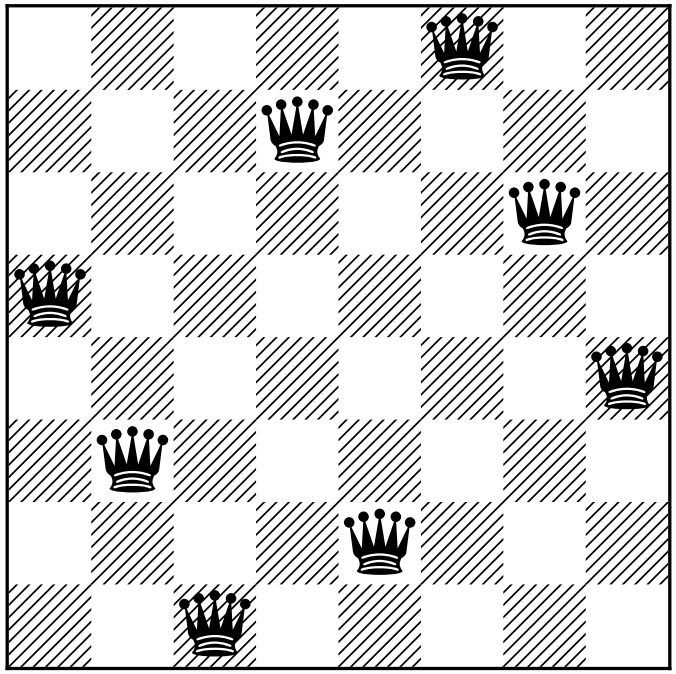
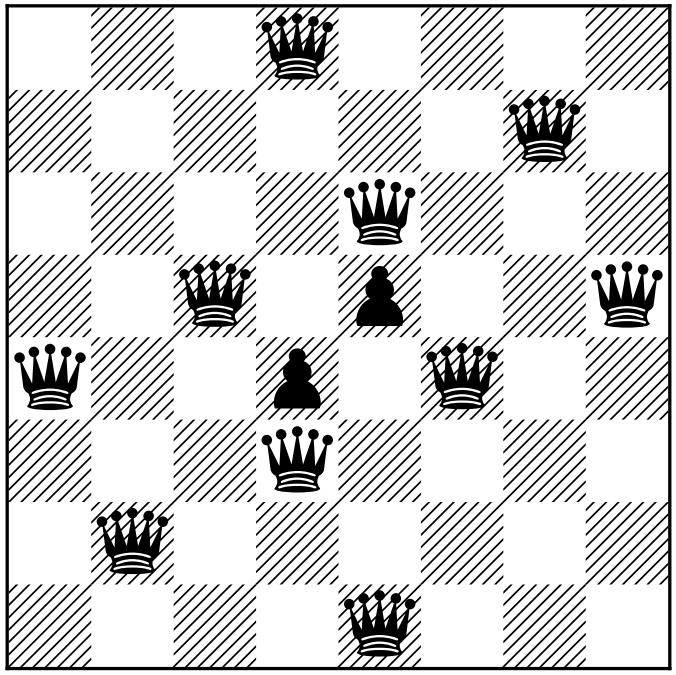
N Queens Problem

- n queens on $n \times n$ chessboard
- no two queens on same row, column, or diagonal



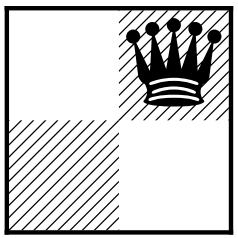
More Than *N* Queens

- pawn between queens in same row, column, or diagonal – pawns block queen attacks

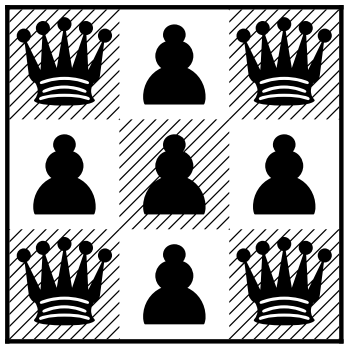


Maximum Queens Problem

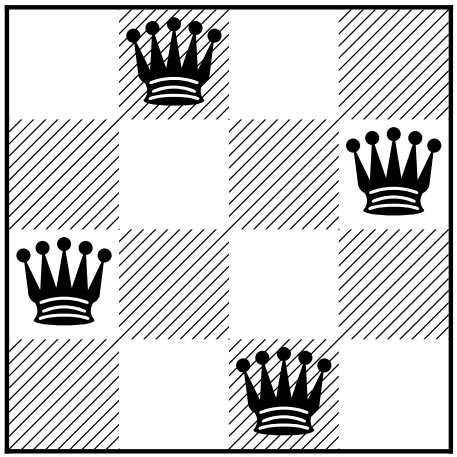
"What is the maximum number of [mutually nonattacking] queens one can place on an $n \times n$ board if one can block as many squares as he needs?" (Zhao, 1998)



1 on 2×2

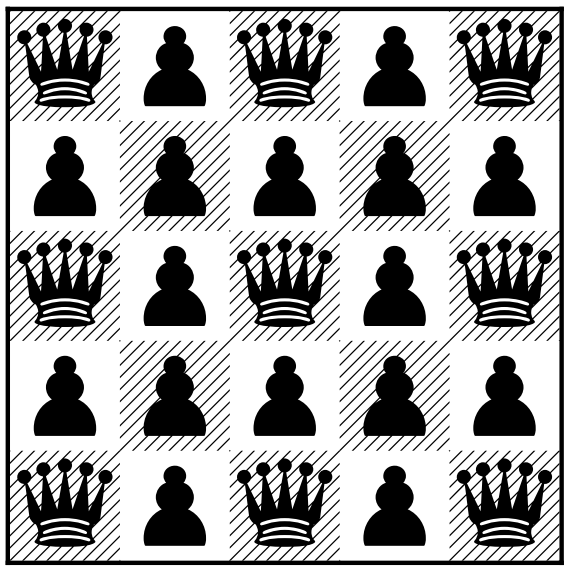


4 on 3×3

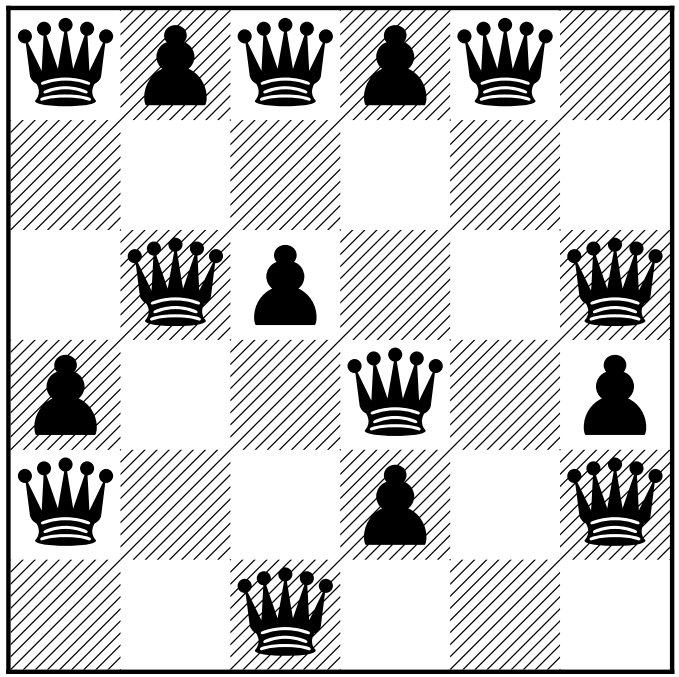


4 on 4×4

More Solutions



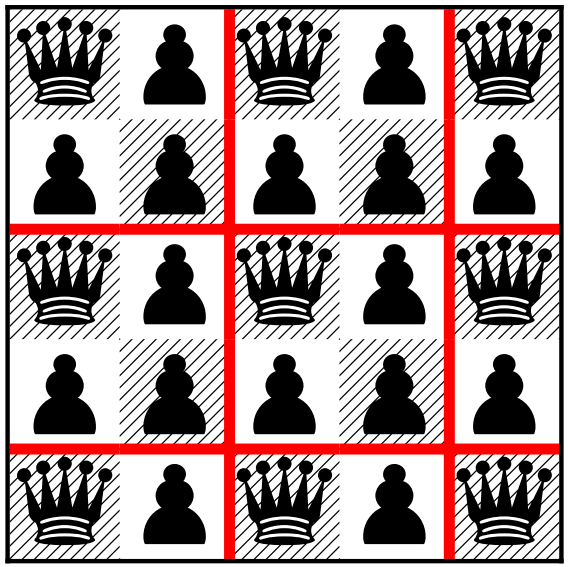
9 on 5×5



9 on 6×6

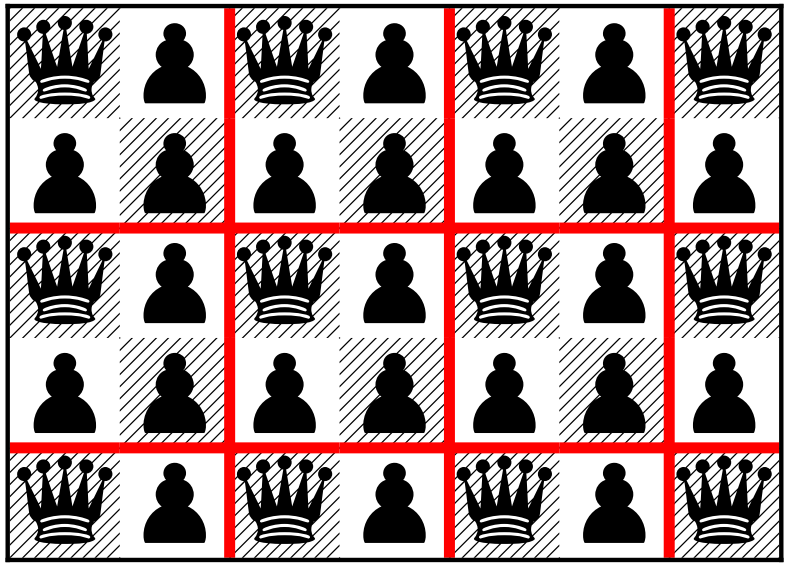
Answer

For an $n \times n$ board, the answer is $\left\lceil \frac{n}{2} \right\rceil^2$.



Answer for Rectangular Board

For an $m \times n$ board, the answer is $\left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$.



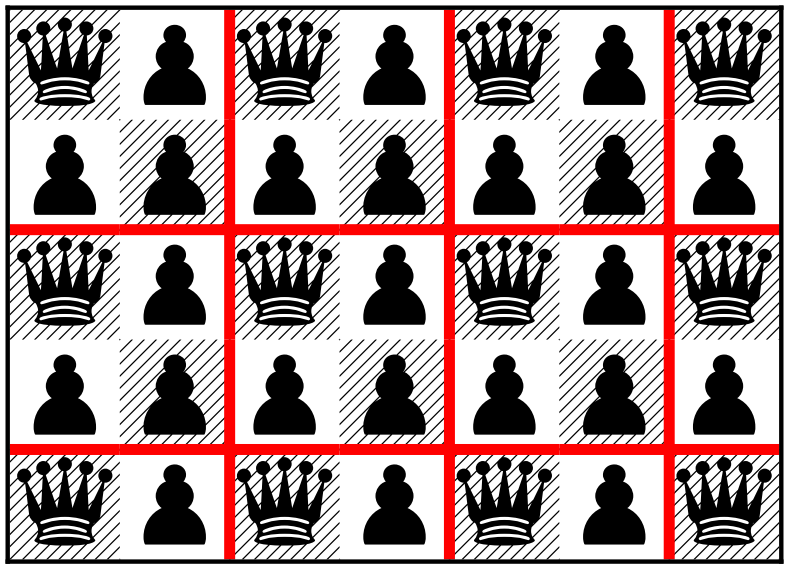
(*c.f.* kings independence number)

Min Pawns for Max Queens Problem

On an $m \times n$ board, how many squares do we need to block in order to place $Q(m,n) := \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ mutually nonattacking queens on the board?

How Many Pawns? (m, n odd)

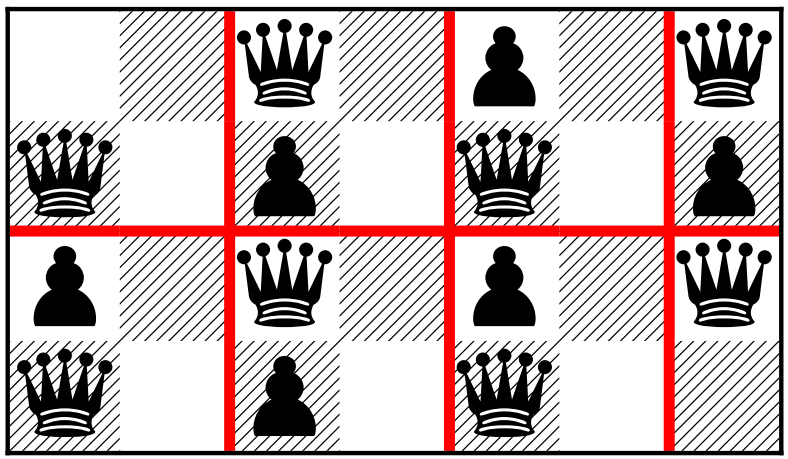
If both m and n odd, just one possible arrangement, with pawns in all squares unoccupied by queens.



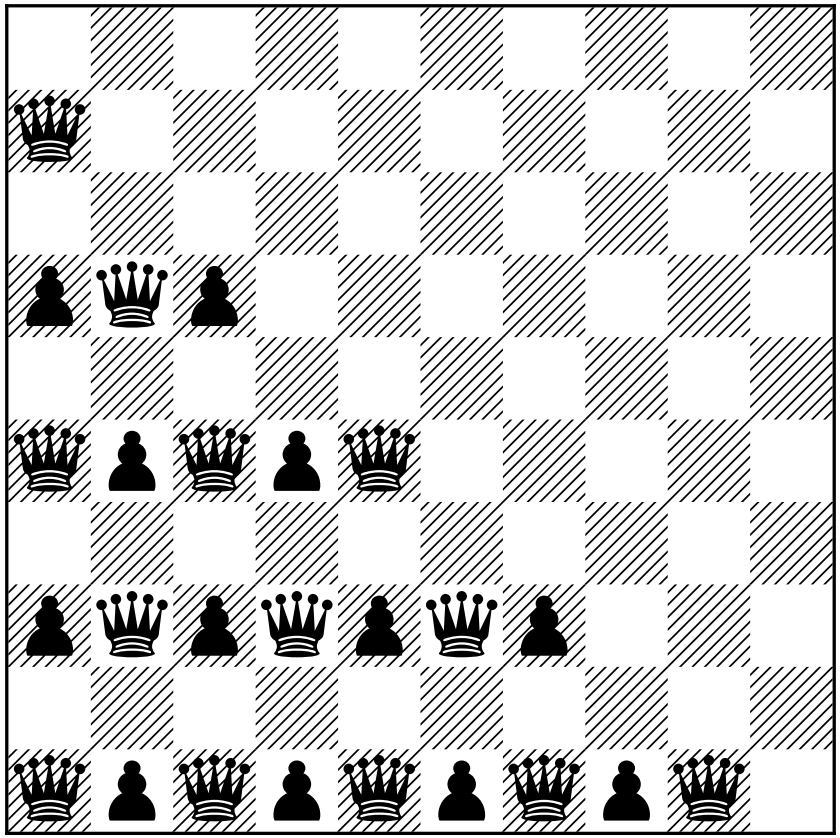
$mn - Q(m,n)$ pawns

How Many Pawns? (m or n even)

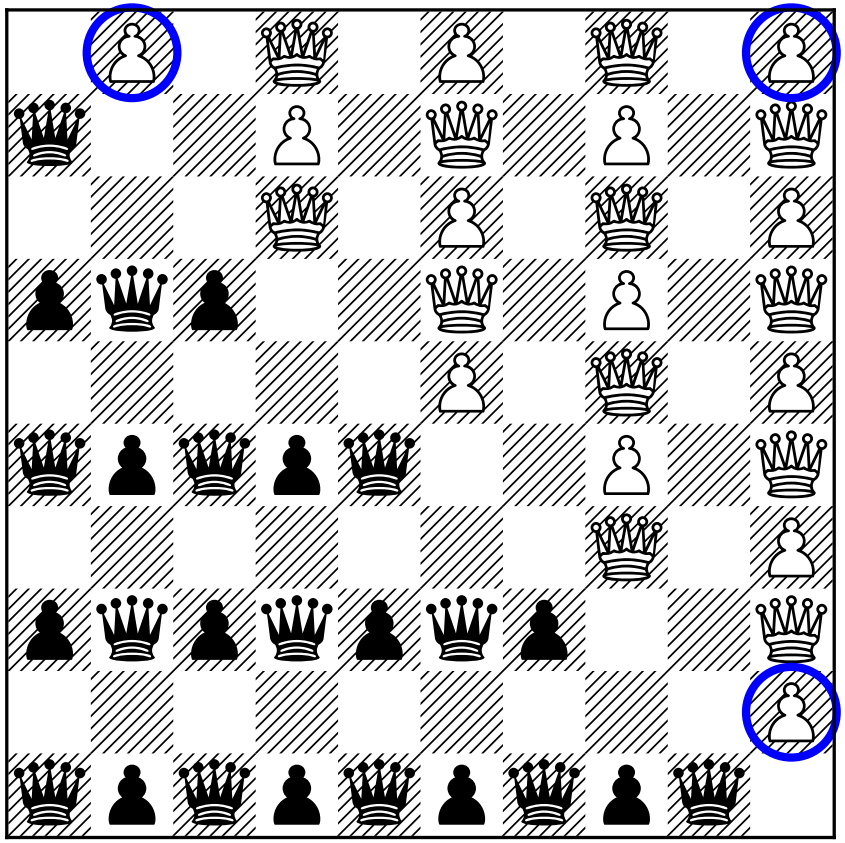
If m or n even, need at most $Q(m,n) - 2$ pawns.



m, n even, $m = n$

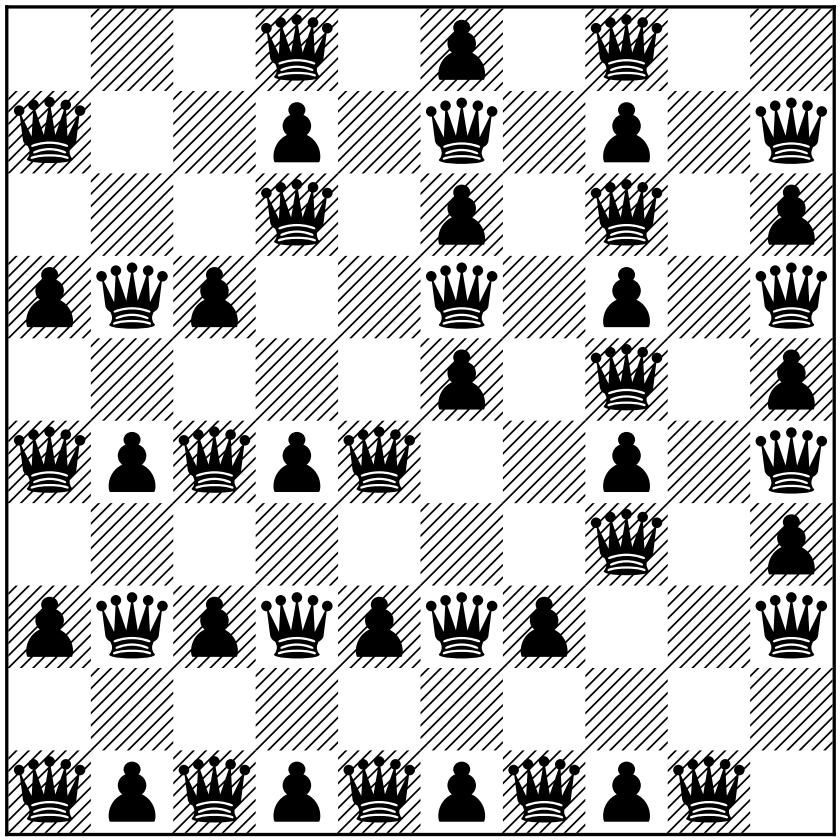


m, n even, $m = n$



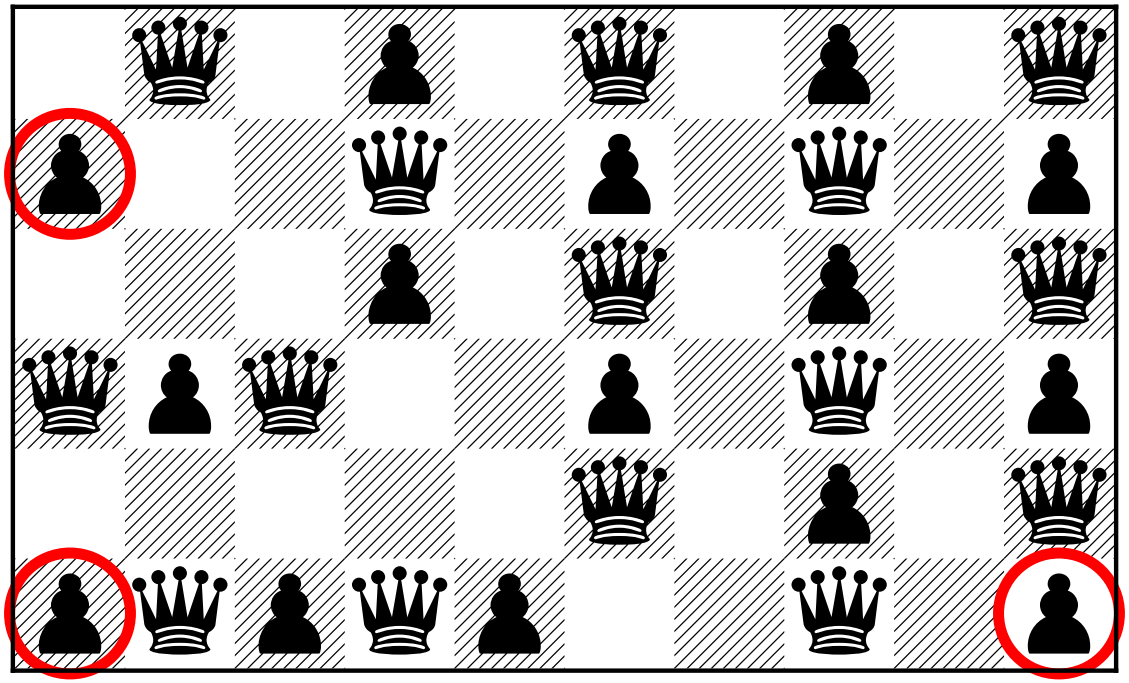
$Q(m,n) - 3$ pawns

m, n even, $m = n$



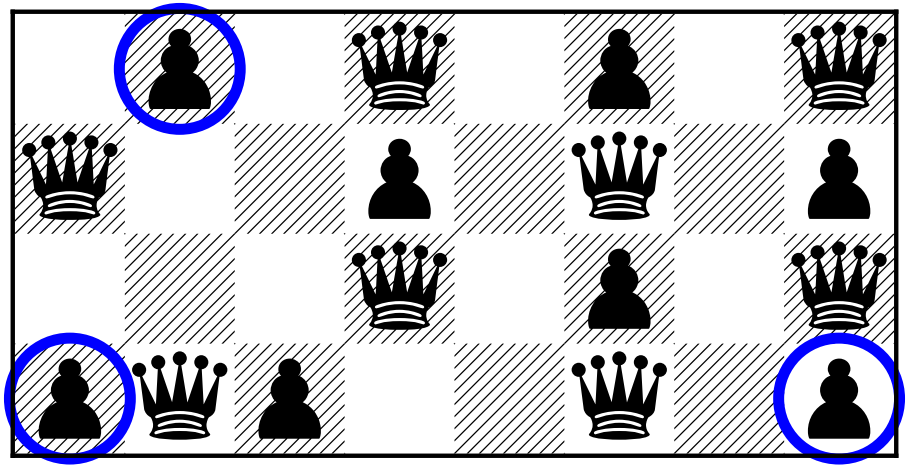
$Q(m, n) - 3$ pawns

m, n even, $m < n$, and $m \equiv 2 \pmod 4$



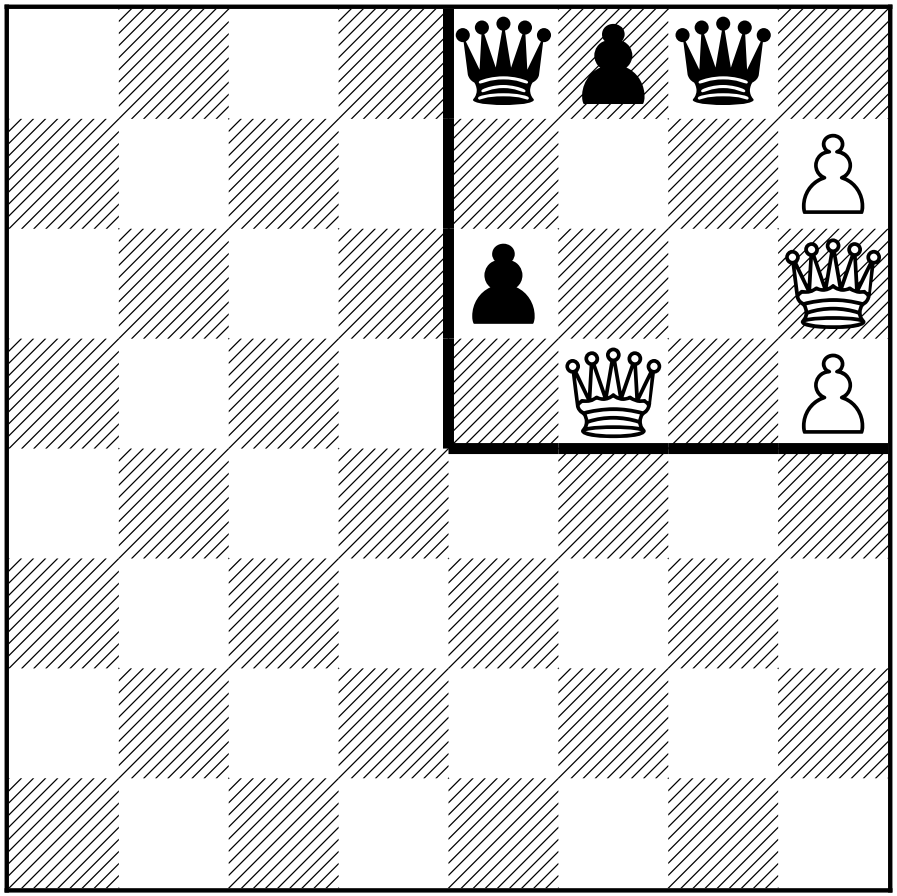
$Q(m,n) - 3$ pawns

m, n even, $m < n$, and $m \equiv 0 (\mathrm{mod}\ 4)$

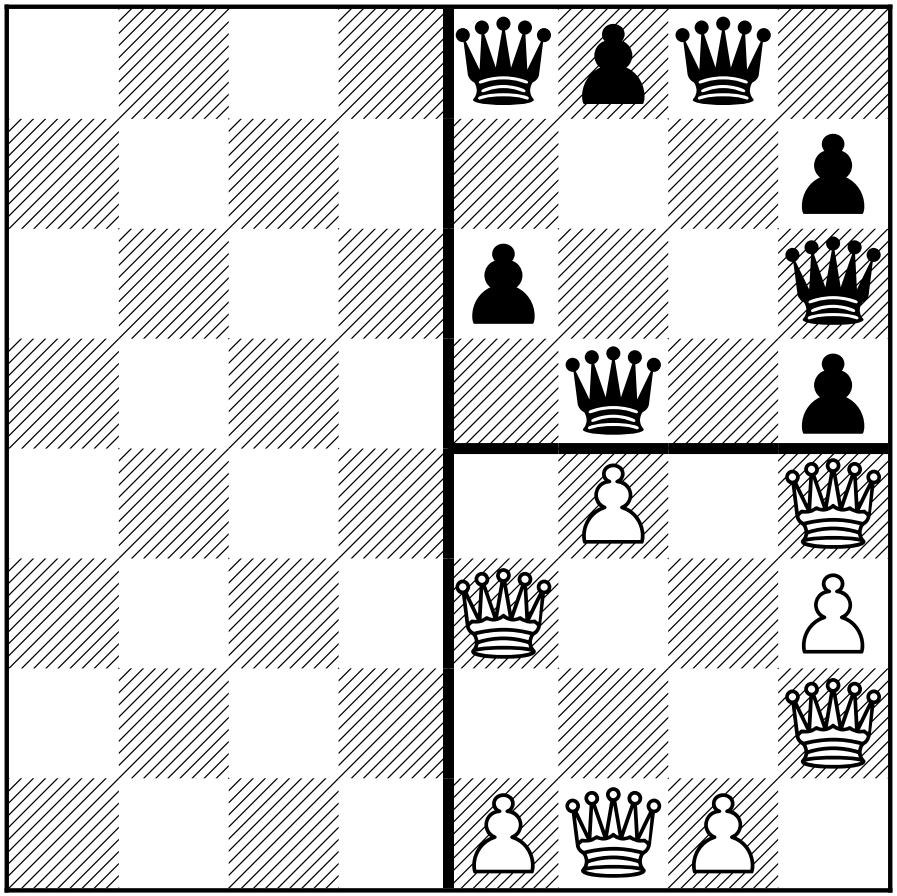


$Q(m, n) - 3$ pawns

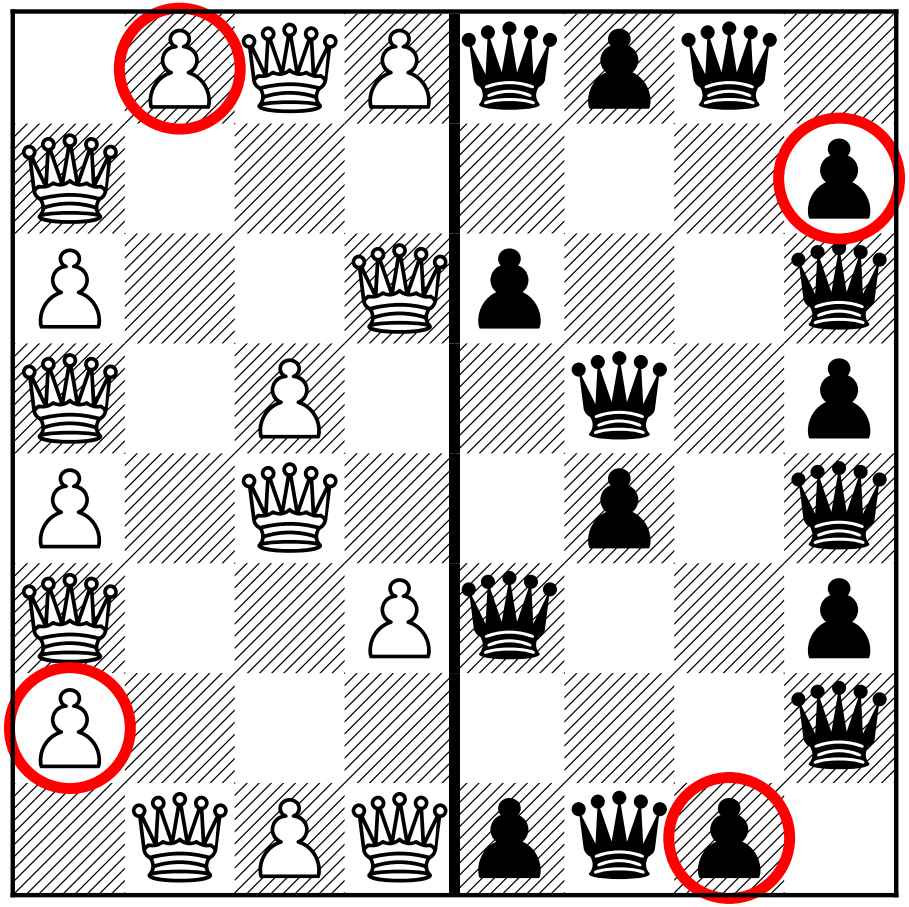
$m = n = 4k, k \geqslant 1$



$m = n = 4k, k \geqslant 1$

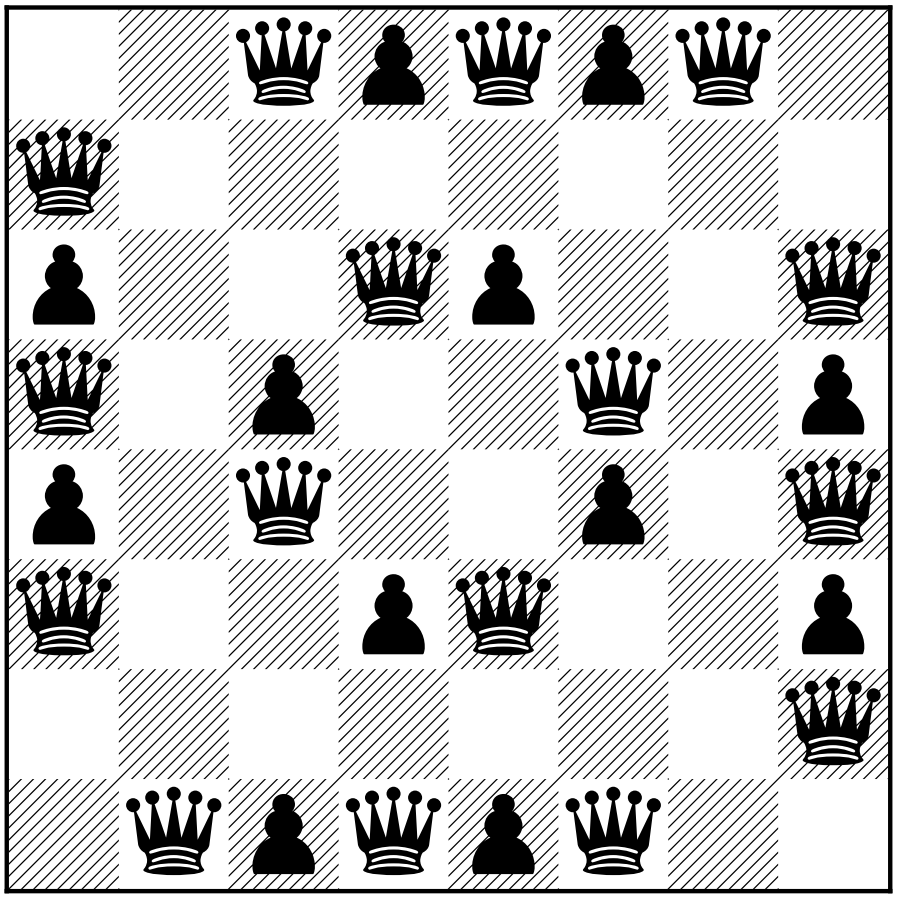


$m = n = 4k, k \geqslant 1$



$Q(m,n) - 4$ pawns

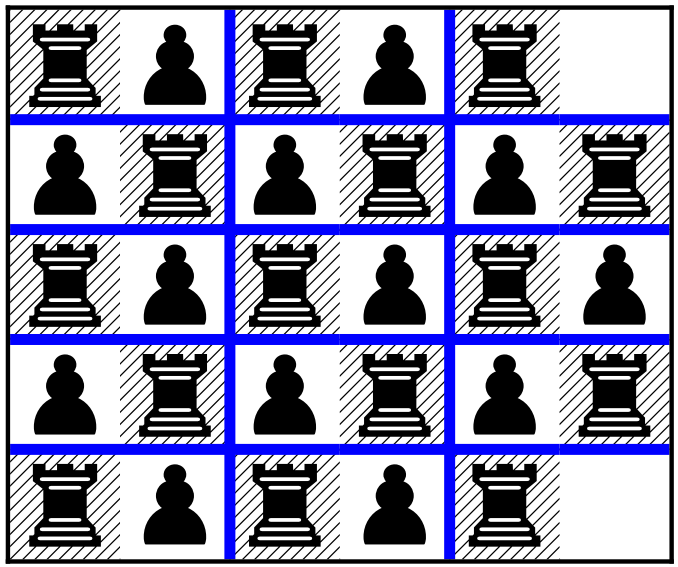
$m = n = 4k, k \geqslant 1$

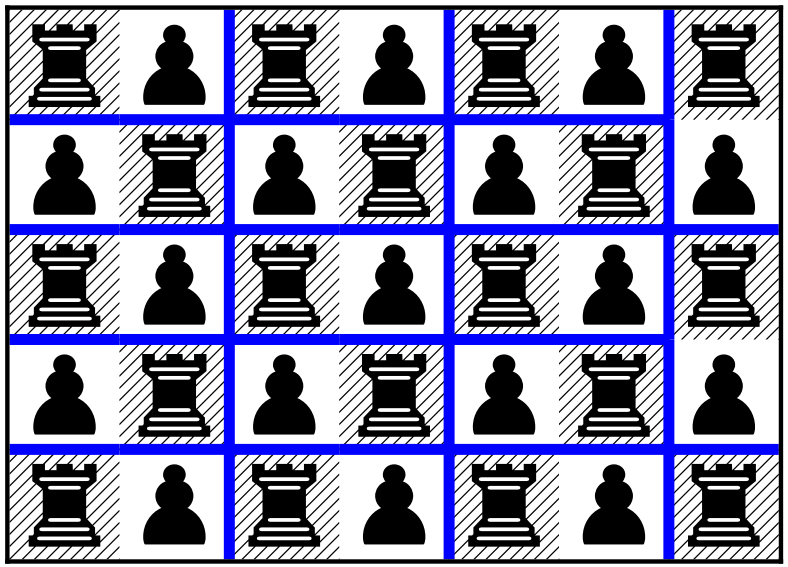


$Q(m,n) - 4$ pawns

Maximum Rooks

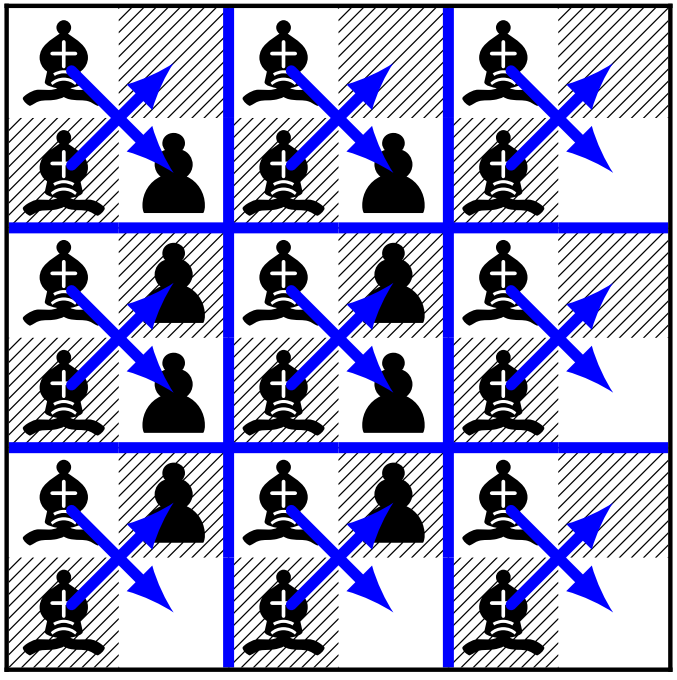
$$R(m,n) = \lceil \frac{mn}{2} \rceil$$





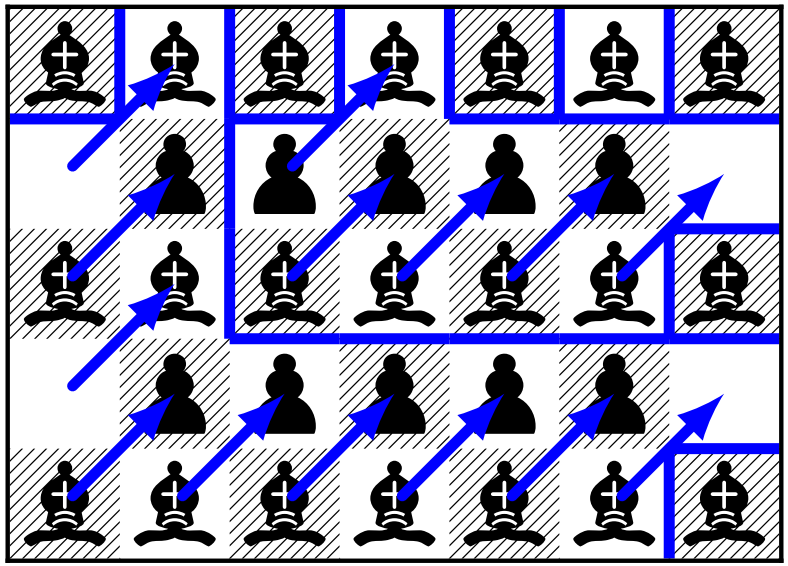
Maximum Bishops (m and n even)

$$B(m,n) = \max\{n\lceil\tfrac{m}{2}\rceil, m\lceil\tfrac{n}{2}\rceil\}$$



Maximum Bishops (m or n odd)

$$B(m,n) = \max\{n\lceil\tfrac{m}{2}\rceil, m\lceil\tfrac{n}{2}\rceil\}$$



Open Problems

- Minimum pawns for m or n even?
- Number of arrangements for m or n even?
- Minimum pawns to place $r < Q(m,n)$ queens?
- Maximum queens for fixed p pawns?
- Other board types (e.g. torus)?

References

- Bell, J. & Stevens, B. (2009). A survey of known results and research areas for n-queens. Discrete Math. 309, no. 1, 1-31.
- Burchett, P. & Chatham, D. Some results for chessboard separation problems. In preparation.
- Watkins, J. J., *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, 2004.

References, Continued

- N+k Queens Problem Pages: http://npluskqueens.info
- NEOS server at http://www.neos-server.org/neos/

Any questions?