The $N - k$ Queens Problem

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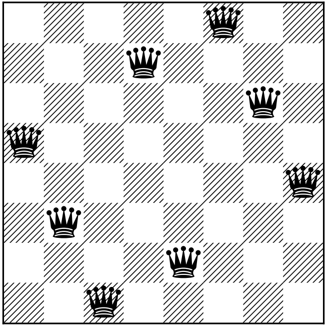
MOVES Conference
New York NY
August 5, 2013

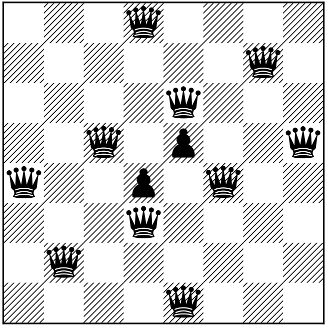
N Queens Problem

- n queens on $n \times n$ chessboard
- no two queens are on same row, column, or diagonal



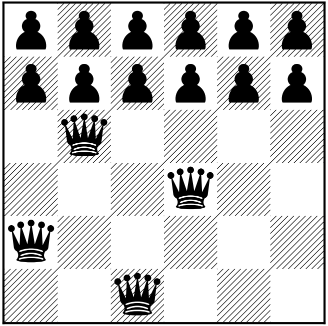
$N + k$ Queens Problem

- $n + k$ queens, k pawns on $n \times n$ chessboard
- pawn between queens in same row, column, or diagonal – pawns block queen attacks



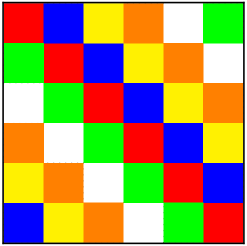
$N - k$ Queens Problem

- pawns do *not* block attacks
- goal: reduce "queens independence number" to $n - k$
- at most nk pawns needed



Sometimes as easy as coloring

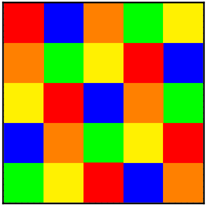
Proposition: To reduce rooks independence number to $n-k$, we need nk pawns.



Proposition: If we can "n-color the queens graph", then to reduce queens independence number to $n-k$, we need nk pawns.

When can we do that?

Proposition (Iyer and Menon, 1966): We can n-color the queens graph for all $n = 6j \pm 1$.

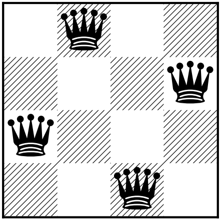


Proposition (Vasquez, 2006): If $n = 6j \pm 1$ and $p = 12, 14, 15, 16, 18, 20, 21, 22, 24, 26, 28, 32$, we can np-color the queens graph on an $np \times np$ board.

However. . .

- infinitely many open cases, starting with $n = 27$
- cannot n-color the queens graph for $n = 2, 3, 4, 6, 8, 9, 10$

$n = 4, k = 1$



Claim: 2 pawns neccessary and sufficient

Hitting Sets

- To reduce independence number below r, need pawns to "hit" every arrangement of r nonattacking queens.
- Can find hitting sets through 0/1 Integer Programming.

0/1 Integer Programming

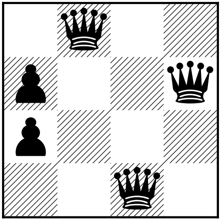
For each board position (i, j), let

$$x_{i,j} = \begin{cases} 1 & \text{if } (i, j) \text{ included,} \\ 0 & \text{otherwise} \end{cases}$$

Minimize $\sum_{i,j} x_{i,j}$ s.t. for each arrangement A of r nonattacking queens,

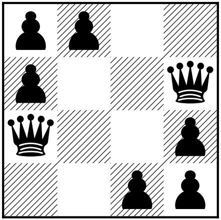
$$\sum_{(i,j) \in A} x_{i,j} \geqslant 1$$

$n = 4, k = 1$



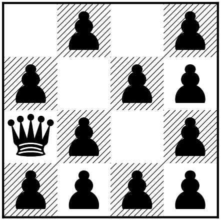
2 pawns

$n = 4, k = 2$



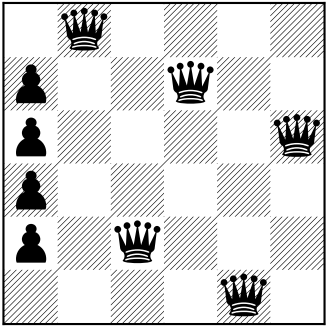
6 pawns

$n = 4, k = 3$



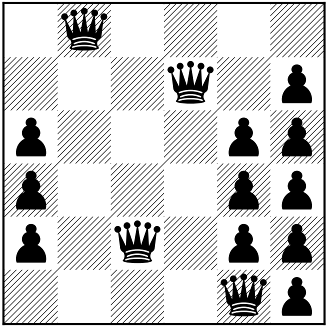
11 pawns

$n = 6, k = 1$



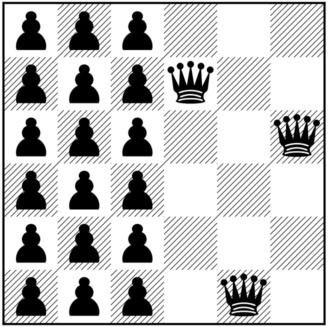
4 pawns

$n = 6, k = 2$



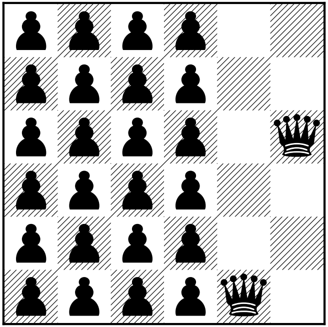
11 pawns

$n = 6, k = 3$



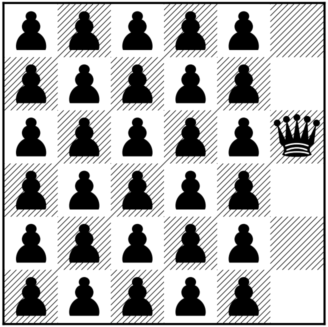
18 pawns

$n = 6, k = 4$



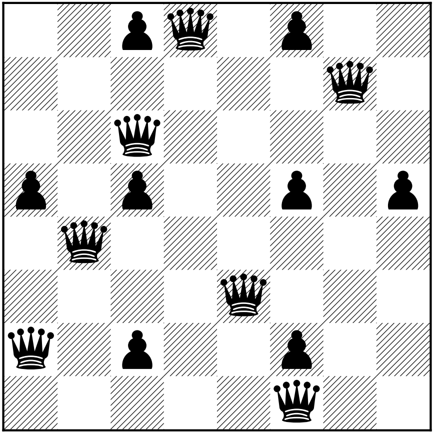
24 pawns

$n = 6, k = 5$



30 pawns

$n = 8, 9, 10, k = 1$



n pawns in each case

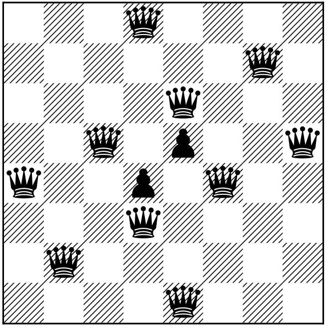
Open Problems

- How many pawns needed?
  Conjecture: For $n \geqslant 7$, to reduce the queens independence number to $n - k$, we need nk pawns.
- How many hitting sets of minimum cardinality?

$n\backslash k$	1	2	3	4	5	6
4	16	12	8	1	-	-
5	120	646	254	32	1	-
6	1296	?	?	?	?	1

Open Problems, continued

- How much difference between blocking and non-blocking pawns?



- Combine with initial n-queens problem?
  - Frustr8tor with Barricade

Open Problems, concluded

- Other pieces and board shapes?

- Reduce other parameters? (domination, total domination, etc.)

References

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- Burchett, P. & Chatham, D. (2013). Some results for chessboard separation problems. Submitted to Util. Math.
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- Chvátal, V. Colouring the queen graphs: http://users.encs.concordia.ca/~chvatal/queengraphs.html
- Fijany, A., & Vatan, F. (2004). New approaches for efficient solution of hitting set problem.
- Vasquez, M. (2006). Coloration des graphes de reines. C. R. Acad. Sci. Paris, Ser. I 342, 157-160.

Any questions?