Queens and Pawns on Square Boards

Handout for the 2015 MOVES Talk
"The Maximum Queens Problem on a Rectangular Board"

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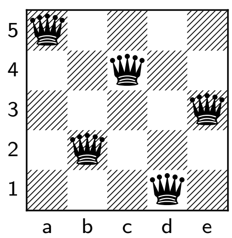
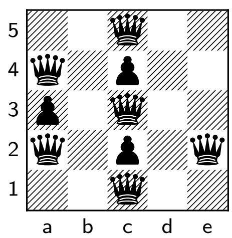
Each diagram below shows the minimum known number of pawns whose placement on the given board will allow the placement of the indicated number of mutually nonattacking queens.

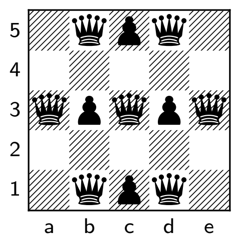
5×5 boards



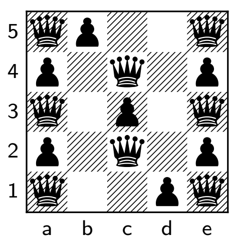
5 queens and 0 pawns



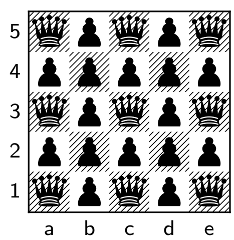
6 queens and 3 pawns



7 queens and 4 pawns

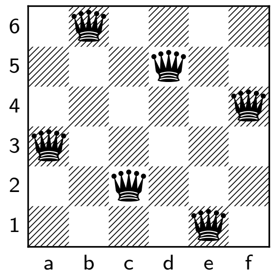


8 queens and 7 pawns

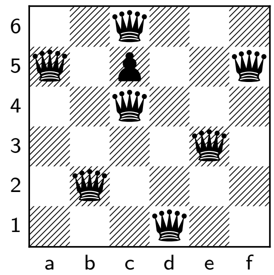


9 queens and 16 pawns

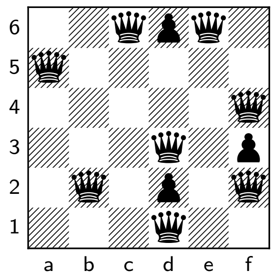
6×6 boards



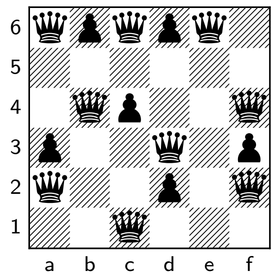
6 queens and 0 pawns



7 queens and 1 pawn

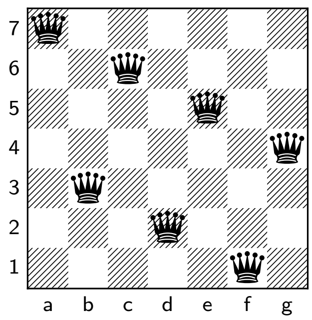


8 queens and 3 pawns

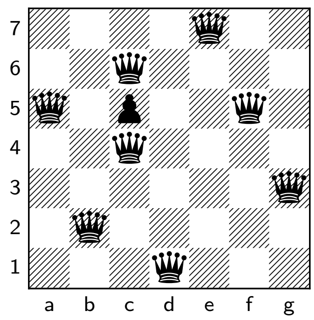


9 queens and 6 pawns

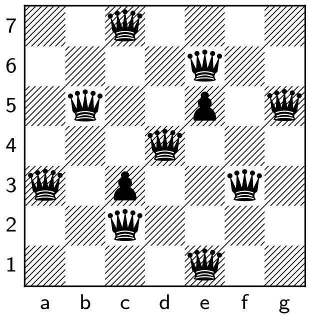
7×7 boards



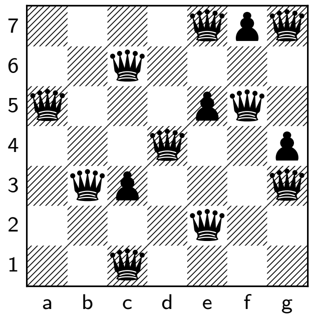
7 queens and 0 pawns



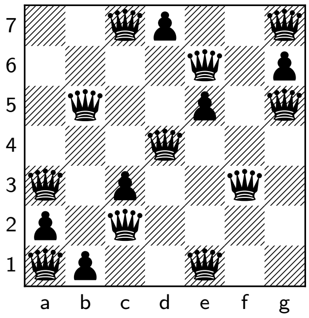
8 queens and 1 pawn



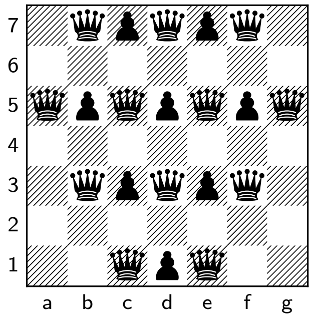
9 queens and 2 pawns



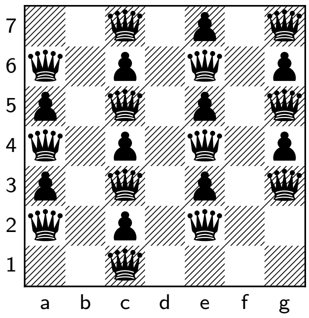
10 queens and 4 pawns



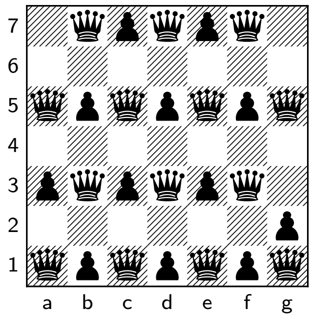
11 queens and 6 pawns



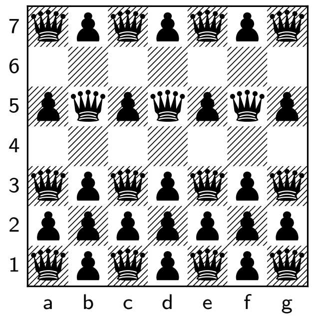
12 queens and 8 pawns



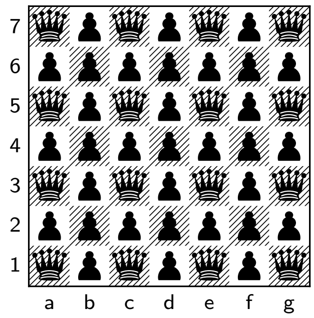
13 queens and 10 pawns



14 queens and 12 pawns

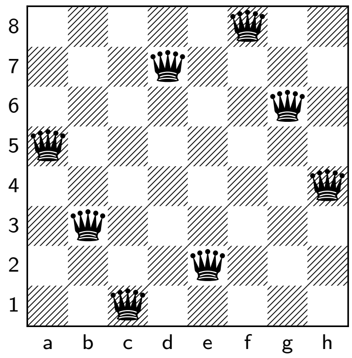


15 queens and 20 pawns

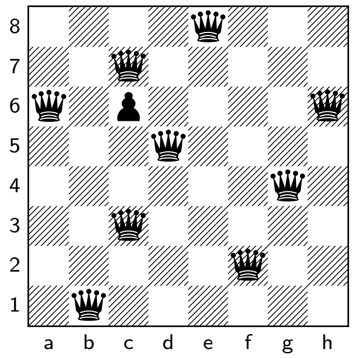


16 queens and 33 pawns

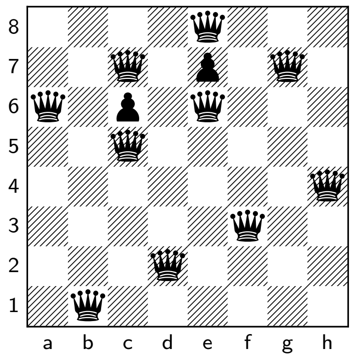
8×8 **boards**



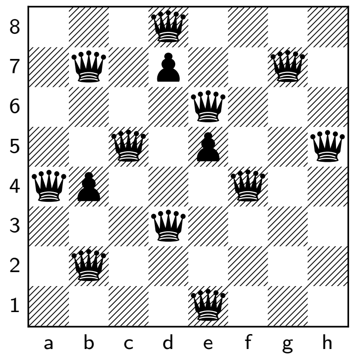
8 queens and 0 pawns



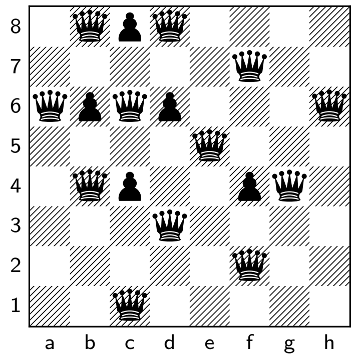
9 queens and 1 pawns



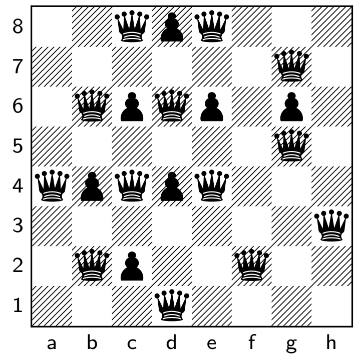
10 queens and 2 pawns



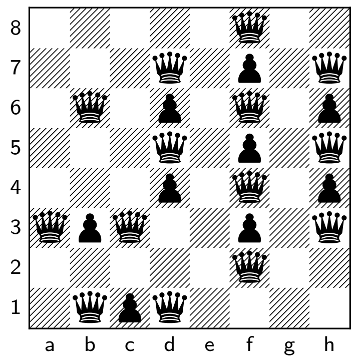
11 queens and 3 pawns



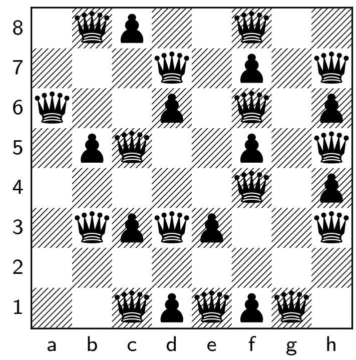
12 queens and 5 pawns



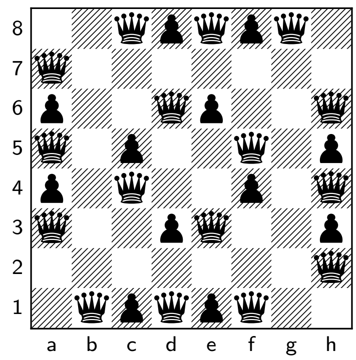
13 queens and 7 pawns



14 queens and 9 pawns



15 queens and 11 pawns



16 queens and 12 pawns

Sources

The n-queens problem

The patterns with no pawns are well-known solutions to the n queens problem. For more information on this problem, see Bell, J. & Stevens, B. (2009). A survey of known results and research areas for n-queens. Discrete Math. 309, no. 1, 1-31.

The $n + k$-queens problem

The patterns where the number of queens equals the order of the board plus the number of pawns (7 queens with 1 pawn on a 6×6 board; 8 queens with 1 pawn and 9 queens with 2 pawns on a 7×7 board; and 9 queens with 1 pawn, 10 queens with 2 pawns, and 11 queens with 3 pawns on an 8×8 board) are examples of solutions to the "$n+k$-queens problem". See http://npluskqueens.info for more information on this problem.

Binary integer programming

To find each of the other patterns in this handout, I formulated the problem of finding the minimum number of pawns needed to allow placement of Q mutually nonattacking queens on a given size board as a binary integer programming problem and submitted the problem to the NEOS server (http://www.neos-server.org/neos/). Here is the formulation of the "minimum pawns for Q queens problem" on an $n \times n$ board:

Suppose we have an $n \times n$ board with rows labeled $0, \ldots, n-1$ and columns labeled $0, \ldots, n-1$.

For $i = 0, \ldots, n-1$ and $j = 0, \ldots, n-1$, let $q_{ij} = 1$ if and only if there is a queen in row i and column j and $q_{ij} = 0$ otherwise. Also, for each i and j, let $p_{ij} = 1$ if and only if there is a pawn in row i and column j and $p_{ij} = 0$ otherwise.

Minimize $\Sigma_{i,j} p_{ij}$ (the number of pawns on the board) subject to the following constraints:

1. $\sum_{i,j} q_{i,j} = Q$. (There are exactly Q queens on the board.)
2. For $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$, $0 \leq q_{ij} + p_{ij} \leq 1$. (A queen cannot occupy the same space as a pawn.)
3. For $0 \leq i \leq n-1$ and $0 \leq j_1 < j_2 \leq n-1$, $\sum_{j=j_1}^{j_2} (q_{ij} - p_{ij}) \leq 1$. (No segment of a row has two queens without a pawn between them.)
4. For $0 \leq j \leq n-1$ and $0 \leq i_1 < i_2 \leq n-1$, $\sum_{i=i_1}^{i_2} (q_{ij} - p_{ij}) \leq 1$. (No segment of a column has two queens without a pawn between them.)
5. a) For $0 < s \leq n-1$ and $0 \leq j_1 < j_2 \leq s$, $\sum_{j=j_1}^{j_2} q_{s-j,j} - p_{s-j,j} \leq 1$.

b) For $n \leq s < 2n-2$ and $s-(n-1) \leq j_1 < j_2 \leq n-1$, $\sum_{j=j_1}^{j_2} q_{s-j,j} - p_{s-j,j} \leq 1$.

(No segment of a "sum diagonal" – the squares (i,j) for which $i+j=s$ for some constant s – has two queens without a pawn between them.)

6. a) For $n-1 > d \geq 0$ and $0 \leq j_1 < j_2 \leq (n-1)-d$, $\sum_{j=j_1}^{j_2} q_{d+j,j} - p_{d+j,j} \leq 1$

   b) For $-1 \geq d > -(n-1)$ and $-d \leq j_1 < j_2 \leq n-1$, $\sum_{j=j_1}^{j_2} q_{d+j,j} - p_{d+j,j} \leq 1$

   (No segment of a "difference diagonal" – the squares (i,j) for which $i-j=d$ for some constant d – has two queens without a pawn between them.)

With a few changes, we get the integer programming formulation of the "minimum pawns for maximum queens problem" on an $m \times n$ board:

Suppose we have an $m \times n$ board with rows labeled $0, \ldots, m-1$ and columns labeled $0, \ldots, n-1$, and $m \leq n$. ($m \leq n$ is an arbitrary choice which makes items 5 and 6 easier to state.)

For $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$, let $q_{ij} = 1$ if and only if there is a queen in row i and column j and $q_{ij} = 0$ otherwise. Also, for each i and j, let $p_{ij} = 1$ if and only if there is a pawn in row i and column j and $p_{ij} = 0$ otherwise.

Minimize $\Sigma_{i,j} p_{ij}$ subject to the following constraints:

1. $\sum_{i,j} q_{i,j} = \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$. (There are exactly $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens on the board.)

2. For $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, $0 \leq q_{ij} + p_{ij} \leq 1$. (A queen cannot occupy the same space as a pawn.)

3. For $0 \leq i \leq m-1$ and $0 \leq j_1 < j_2 \leq n-1$, $\sum_{j=j_1}^{j_2} (q_{ij} - p_{ij}) \leq 1$. (No segment of a row has two queens without a pawn between them.)

4. For $0 \leq j \leq n-1$ and $0 \leq i_1 < i_2 \leq m-1$, $\sum_{i=i_1}^{i_2} (q_{ij} - p_{ij}) \leq 1$. (No segment of a column has two queens without a pawn between them.)

5. a) For $0 < s \leq m-1$ and $0 \leq j_1 < j_2 \leq s$, $\sum_{j=j_1}^{j_2} q_{s-j,j} - p_{s-j,j} \leq 1$.

   b) For $m \leq s < m+n-2$ and $s-(m-1) \leq j_1 < j_2 \leq n-1$, $\sum_{j=j_1}^{j_2} q_{s-j,j} - p_{s-j,j} \leq 1$.

   (No segment of a "sum diagonal" – the squares (i,j) for which $i+j=s$ for some constant s– has two queens without a pawn between them.)

6. a) For $m-1 > d \geq 0$ and $0 \leq j_1 < j_2 \leq (n-1)-d$, $\sum_{j=j_1}^{j_2} q_{d+j,j} - p_{d+j,j} \leq 1$

   b) For $-1 \geq d > -(m-1)$ and $-d \leq j_1 < j_2 \leq n-1$, $\sum_{j=j_1}^{j_2} q_{d+j,j} - p_{d+j,j} \leq 1$

   (No segment of a "difference diagonal" – the squares (i,j) for which $i-j=d$ for some constant d – has two queens without a pawn between them.)