Separation Numbers of Chessboard Graphs

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Acknowledgments

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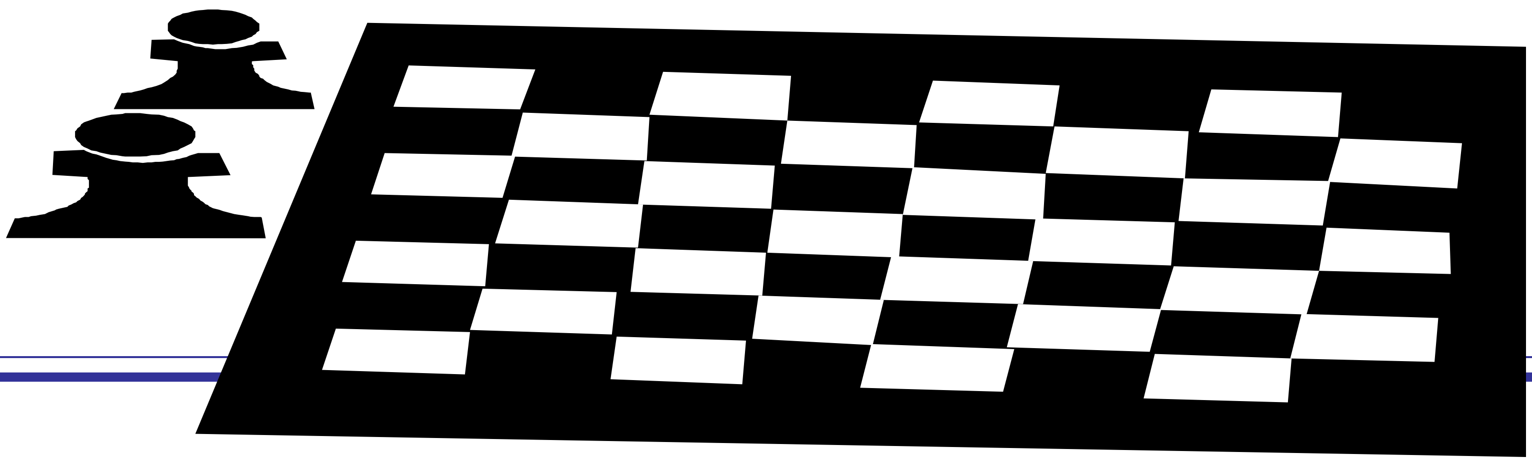
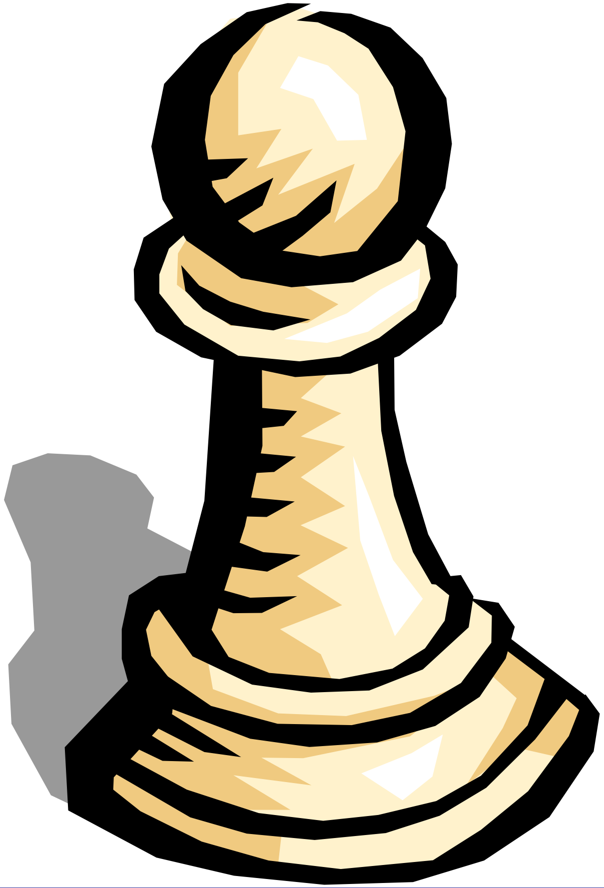


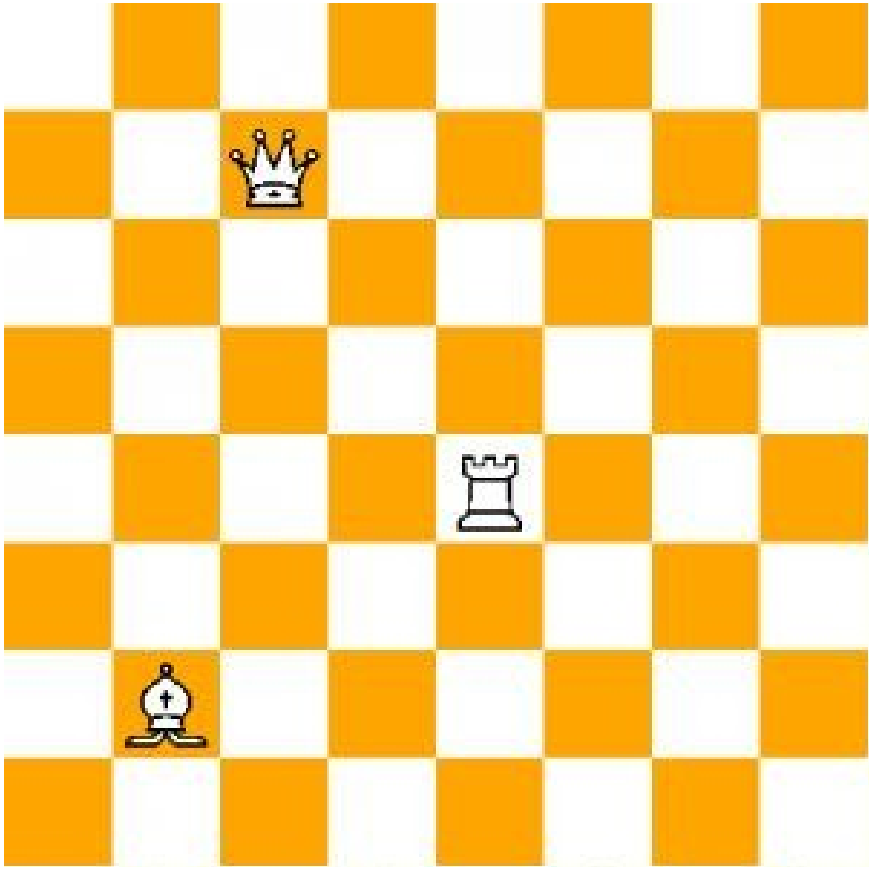
Outline

- Preliminaries
- Independence Separation
- Domination Separation
- Open Problems
- References



Preliminaries: Chess piece moves

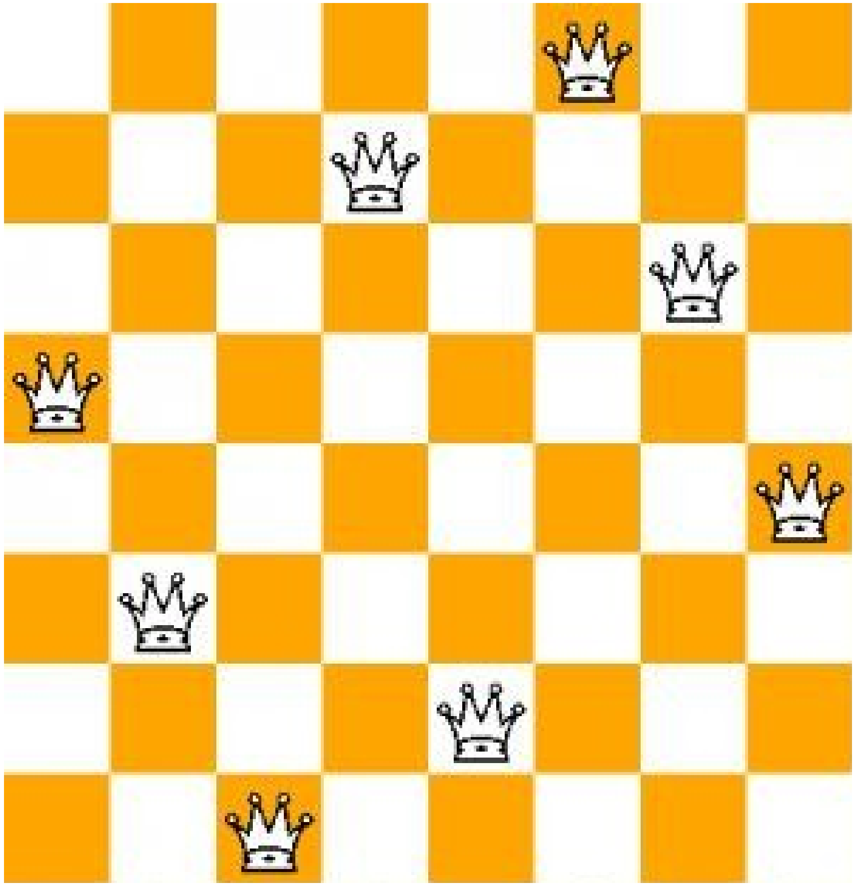
- Queen: any number of spaces vertically, horizontal, or diagonally
- Rook: any number of spaces vertically or horizontally
- Bishop: any number of spaces diagonally



Preliminaries: Chessboard graphs

- Vertices: Squares of a n-by-n chessboard
- Queens graph: *ab* is an edge iff a Queen could move from *a* to *b* in 1 move
- Rooks graph: *ab* is an edge iff a Rook could move from *a* to *b* in 1 move
- Bishops graph: *ab* is an edge iff a Bishop could move from *a* to *b* in 1 move

The Eight Queens Problem



- Place eight queens on a standard chessboard so that no two attack each other.
- First posed in 1848.
- Generalized to N queens on an N x N board.
- Independence number

More about N-queens…

- Theorem: For N > 3, there is at least one solution to the N-queens problem.
- Proved first by Ahrens in 1910.
- Also proved by Hoffman, Loessi, and Moore in 1969. (*Mathematics Magazine*, March-April 1969, 66-72.)
- Also proved by others.

The More-than-N-Queens Problem

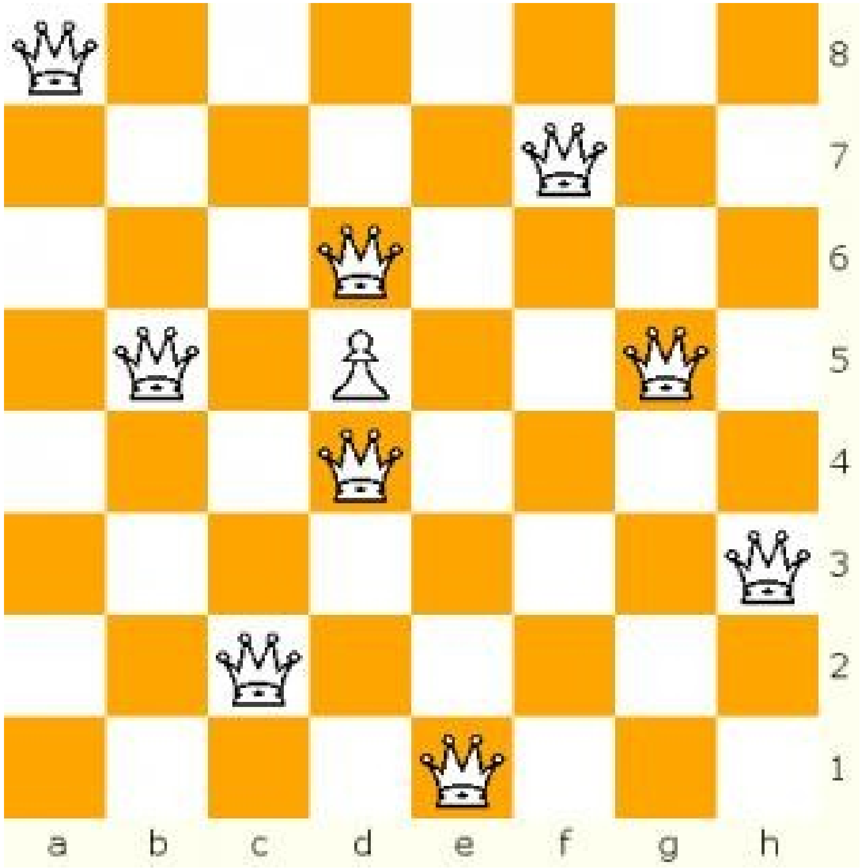
- Remove as few squares as possible to allow more than N queens on an N-by-N board.
- 1995 – Michael Anshel, AI class, CUNY
- 1998 – Kaiyan Zhao, “The Combinatorics of Chessboards” Ph.D. thesis, CUNY.

The Nine Queens Contest

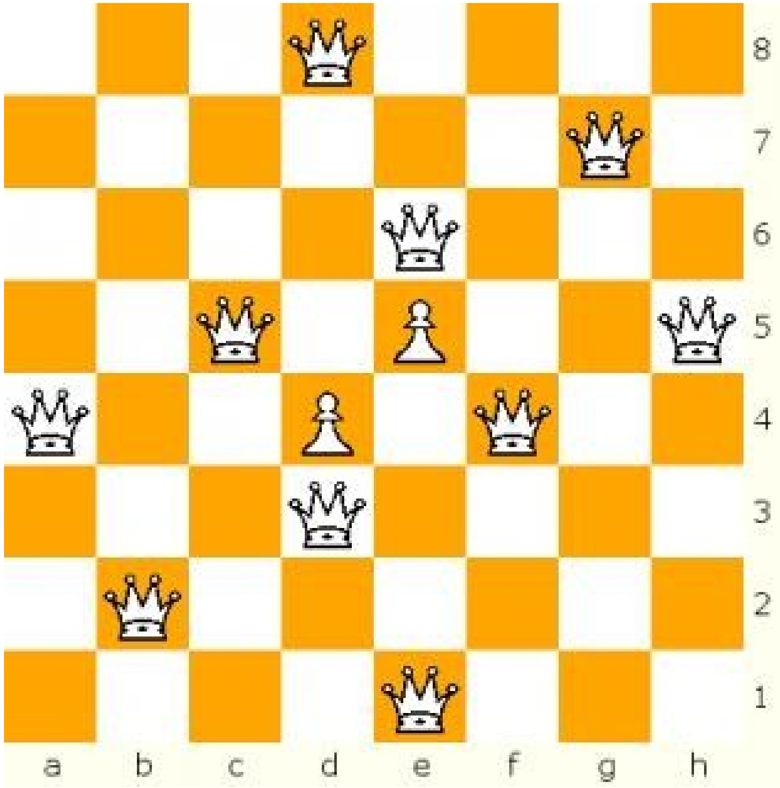
- January-March 2004, The Chess Variant Pages at chessvariants.org.
- If we place a pawn between two queens on the same row (or column or diagonal), the queens no longer attack each other.
- Question: How many pawns do we need in order to put 9 nonattacking queens on a standard chessboard?

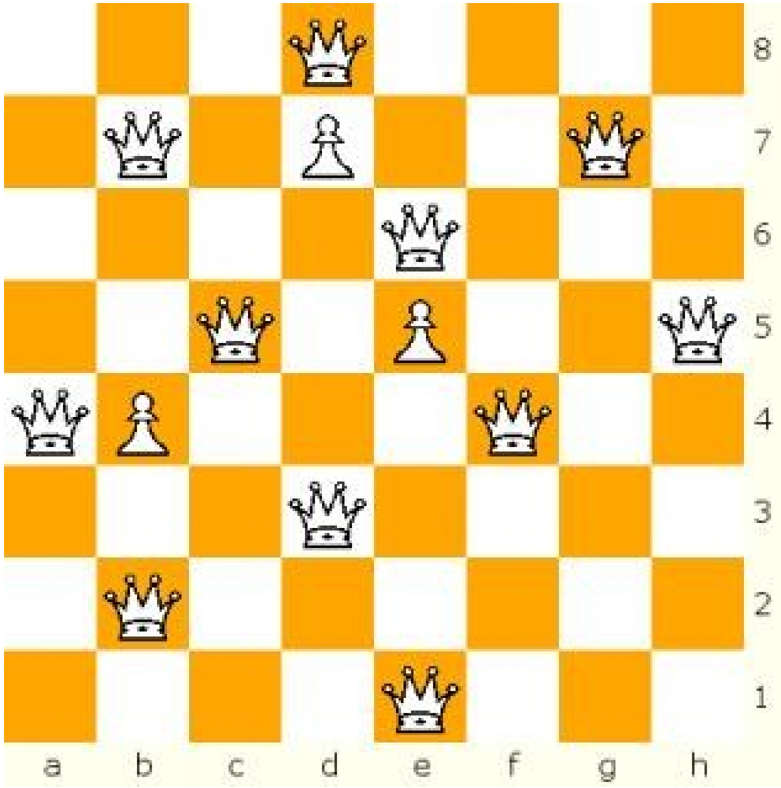
Solution to 9 Queens Contest

- Answer: One pawn.
- A solution:
  - QUEENS at a8, b5, c2, d4, d6, e1, f7, g5, h3
  - PAWN at d5



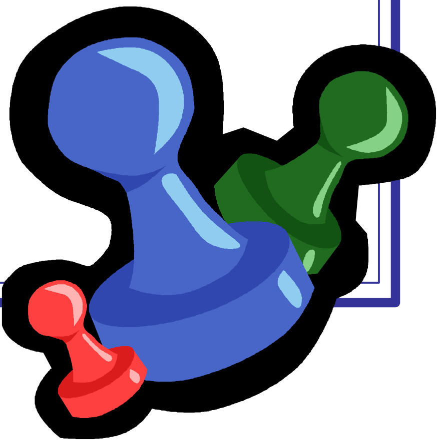
Why stop at nine?





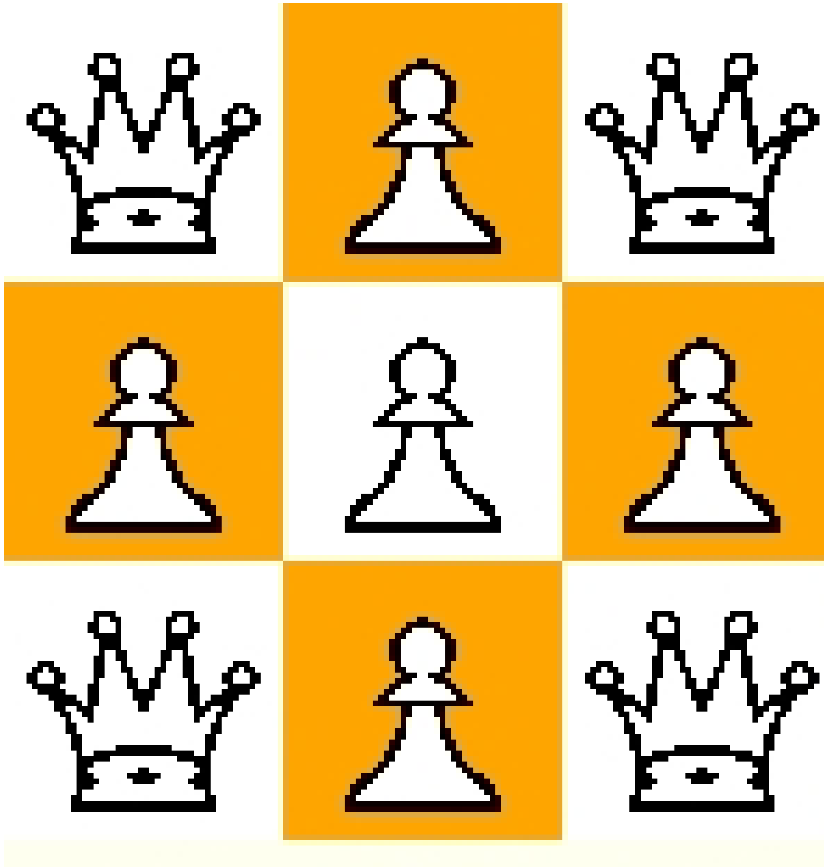
Independence Separation Numbers

- **$s_Q(\beta,k,n)$** is the minimum number of Pawns we need to place on an n-by-n chessboard so that the Queens graph on the remaining squares has **independence** number k.
- Similar definition for other pieces.

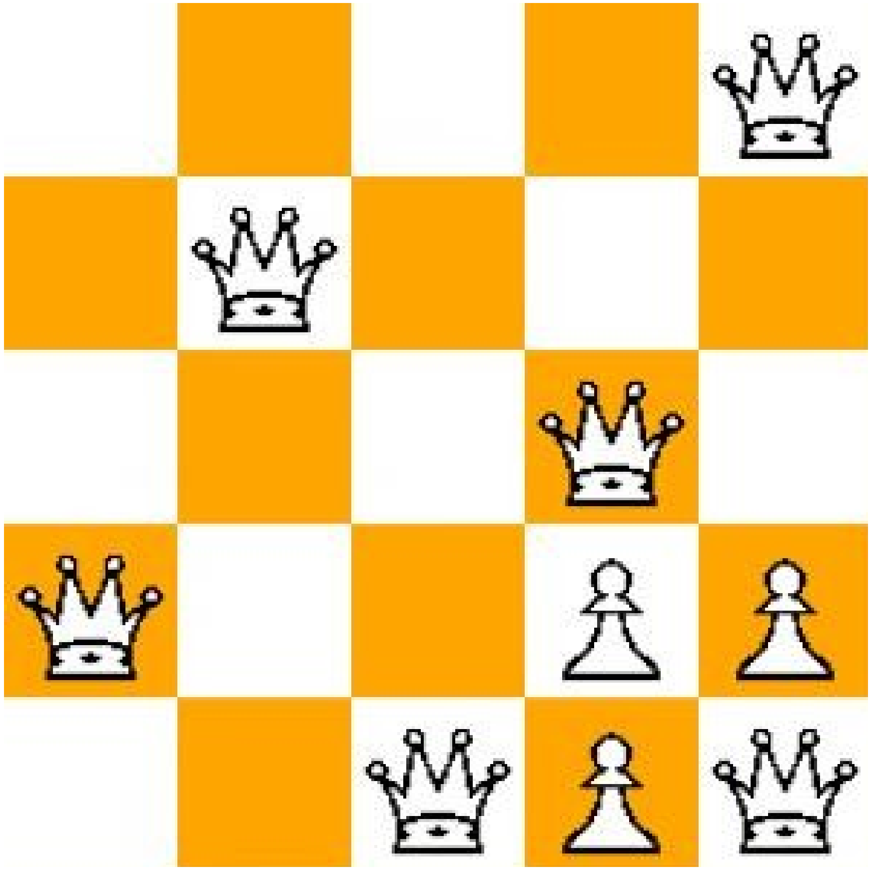


Independence Separation: Adding One Queen

- $s_Q(\beta,4,3) = 5$
- $s_Q(\beta,5,4)$ does not exist, since when we put 5 queens on a 4 x 4 board, at least two queens will be on adjacent squares.



Adding One Queen (p. 2)

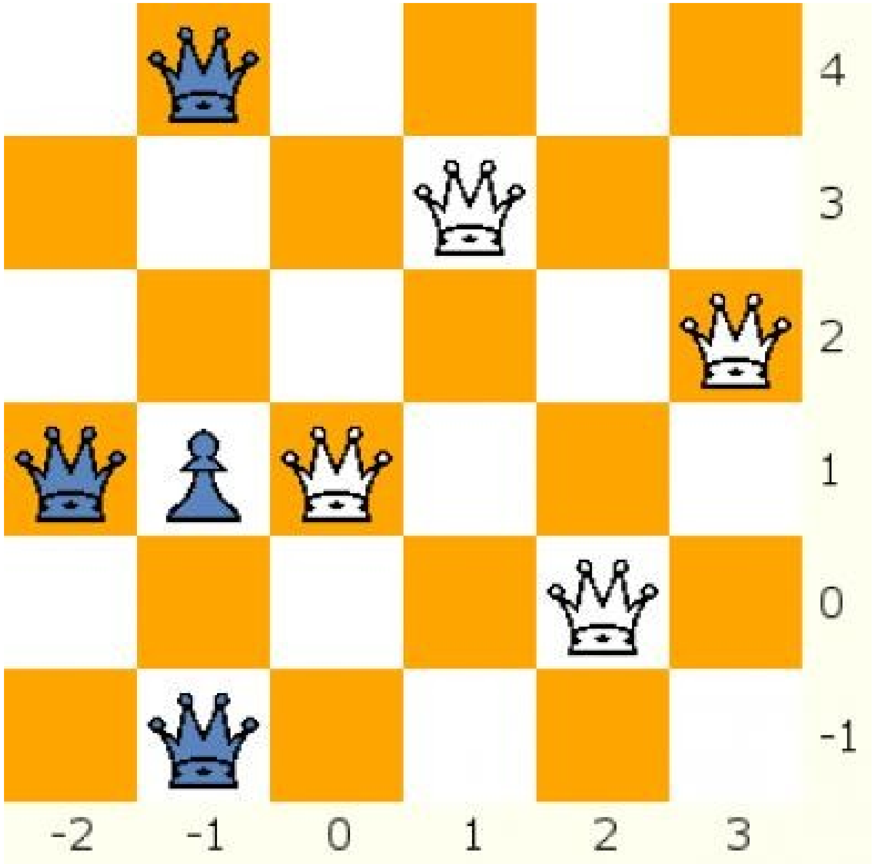


- $s_Q(\beta,6,5) = 3$
  - K. Zhao (1998)
- **Theorem 1**:

  For n > 5,
  $s_Q(\beta,n+1,n) = 1$.

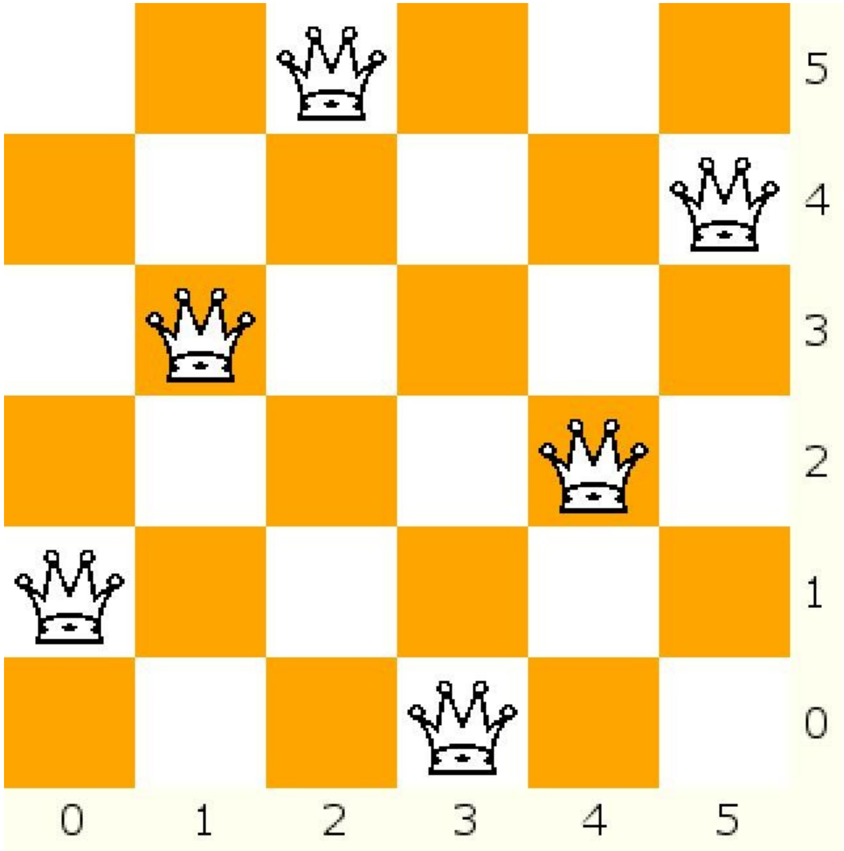
Sketch of Proof of Theorem 1

- We take known solutions to the n-queens problem and add extra rows, columns, queens, and a pawn.



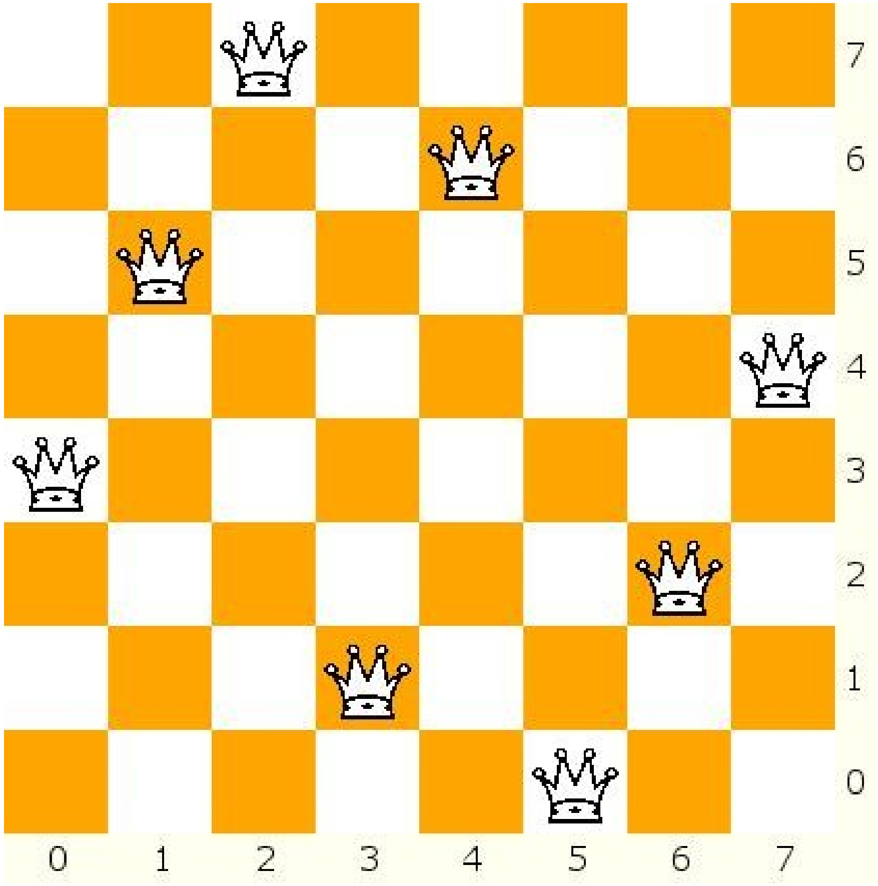
N-Queens Construction A

- $n \geq 4$
- $n \equiv 0$ or 4 (mod 6)
- Number rows and columns 0, 1,…, n-1
- Queens at (2i+1,i) for i=0,…, n/2-1
- Queens at (2i-n,i) for i=n/2,…,n-1.



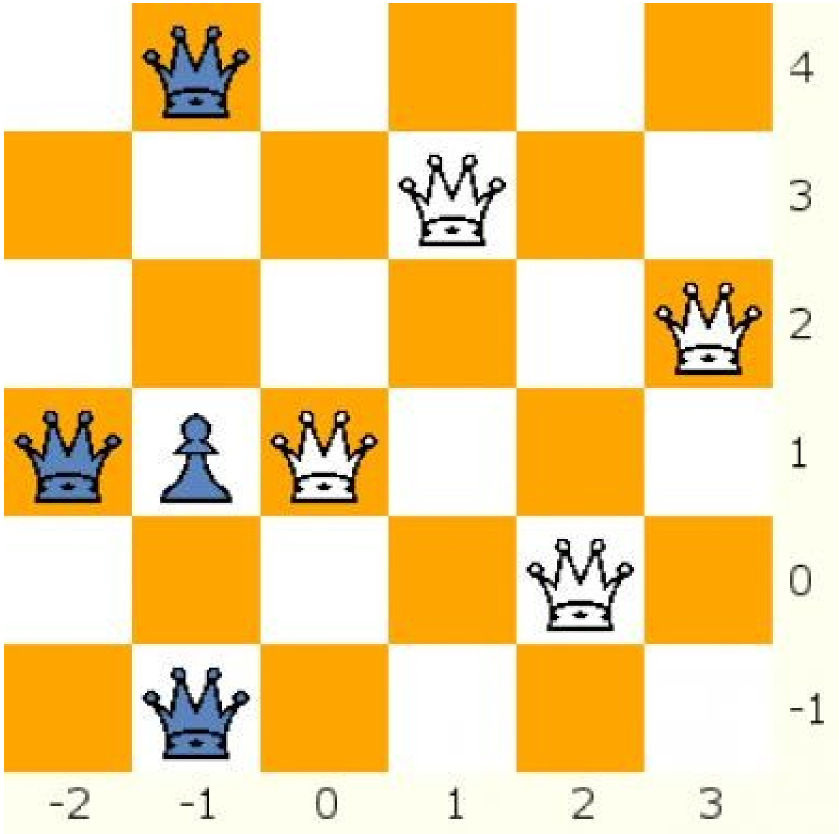
N-Queens Construction B

- $n \geq 4$
- $n \equiv 2$ or $4 \pmod 6$
- Number rows and columns 0, 1,…, n-1
- Queens at (n/2+2i-1 (mod n),i) for i=0,1,…n/2-1 and at (n/2+2i+2 (mod n), i) for i=n/2-1,…,n-1



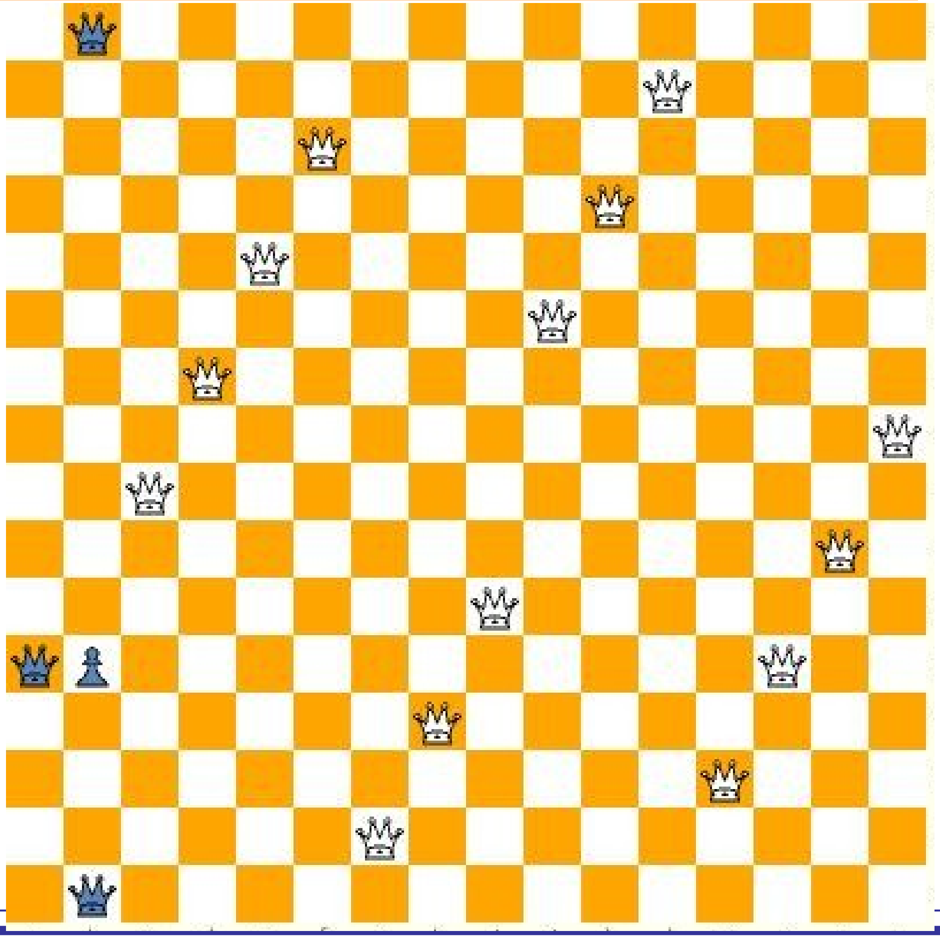
Pattern I: $N \geq 6$, $N \equiv 0$ or 2 (mod 6)

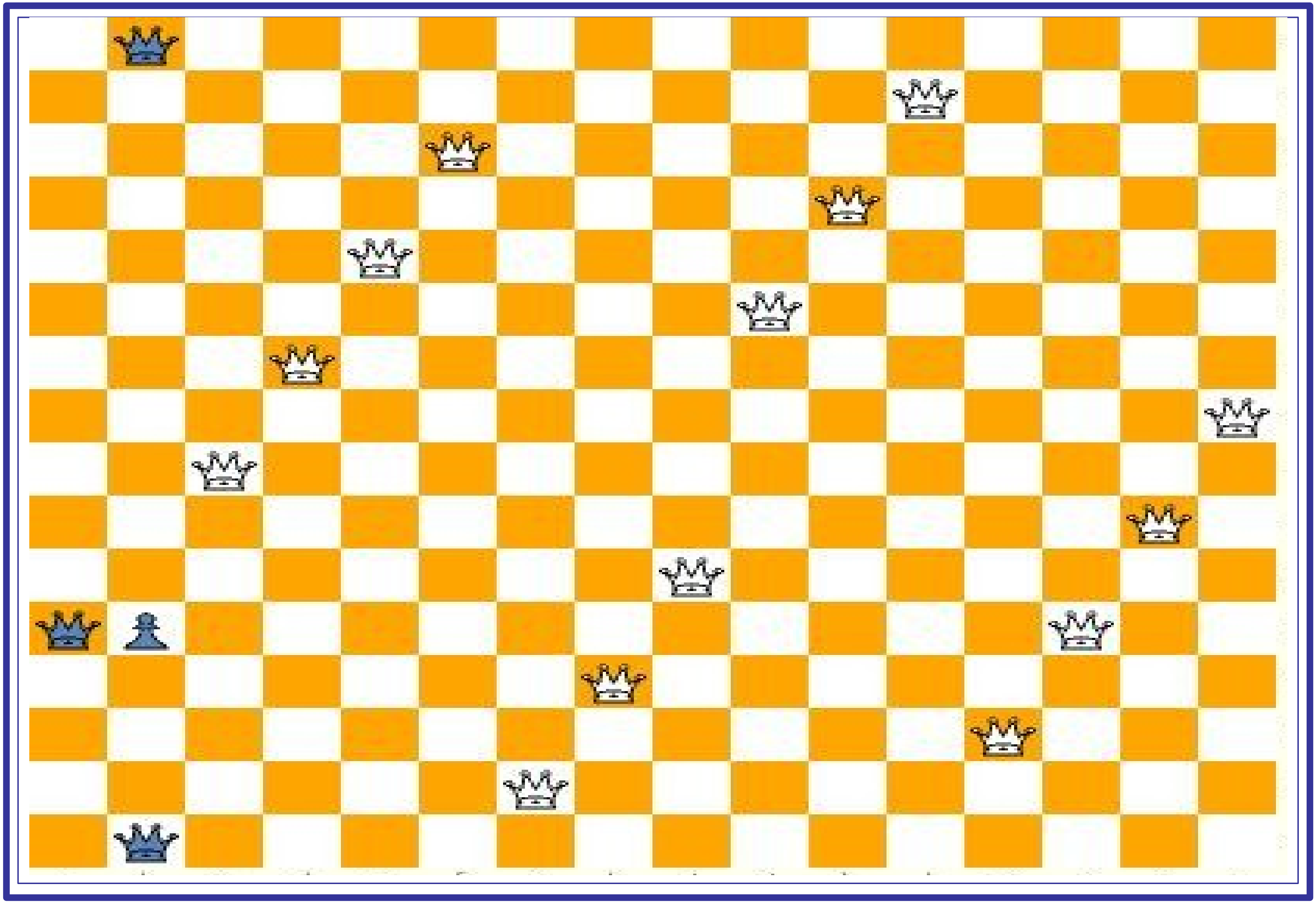
- Take Construction A solution to (n-2)-Queens.
- Add two columns to left and one row to top and bottom
- Put pawn at ((n-2)/2 –1, -1).
- Put extra queens at top and bottom of the pawn's column and to the left of the pawn.



Pattern II: $N \geq 10$, $N \equiv 0$ or 4 (mod 6)

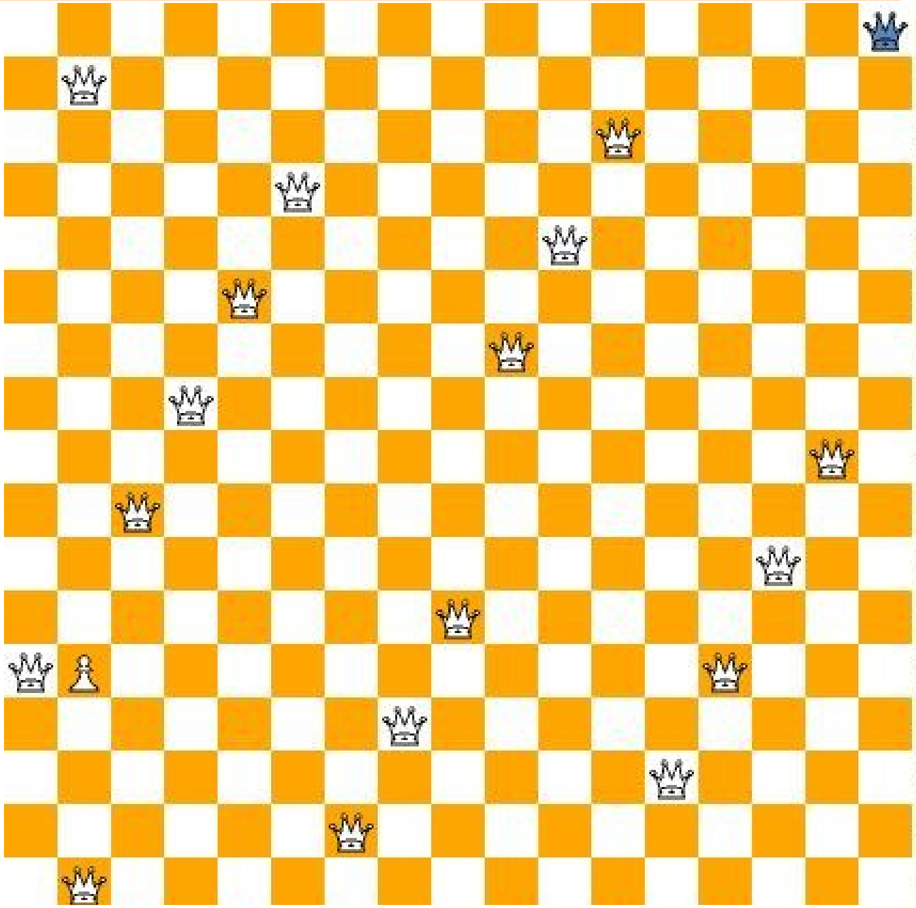
- Take Construction B solution to (N-2)-Queens. Add rows and columns as in Pattern I.
- If N=10, w=4.
- If N=12, w=7.
- If N>12, w=$\lfloor (N-1)/4 \rfloor$
- Pawn: (w, -1)
- Extra Queens: (w, -2), (-1,-1), (N-2, -1)

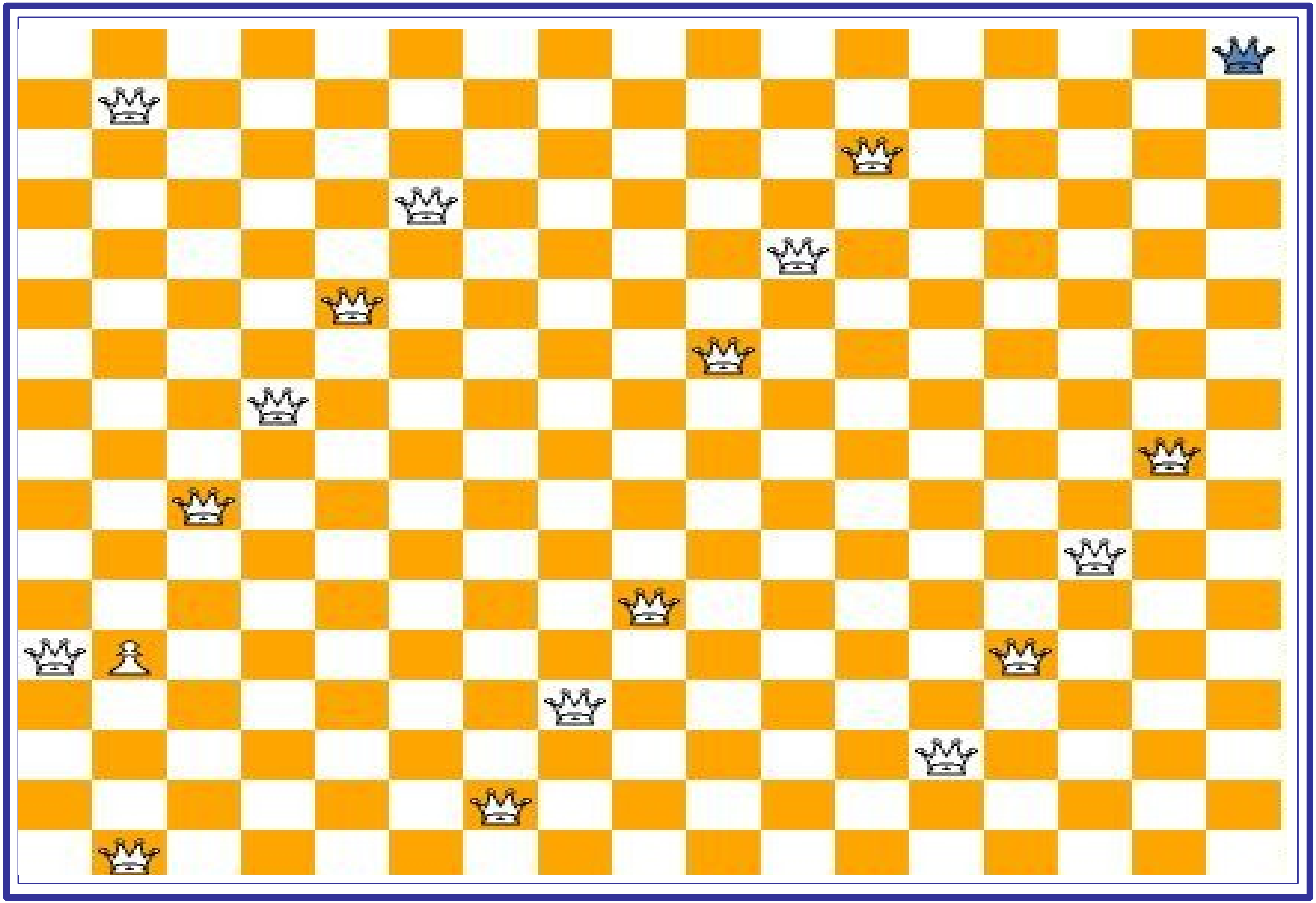




Pattern III: $N \geq 11$, $N \equiv \pm 1 \pmod 6$

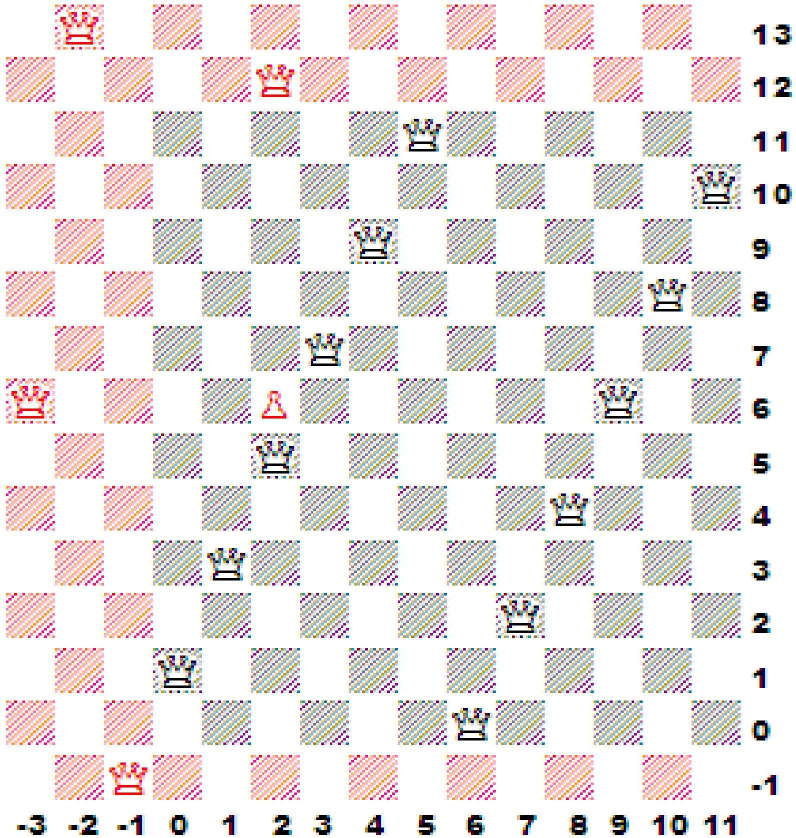
- Add row on top and column to right of Pattern II solution to (N-1)-Queens.
- Pattern II leaves main diagonal open, so add a queen to the upper right corner.



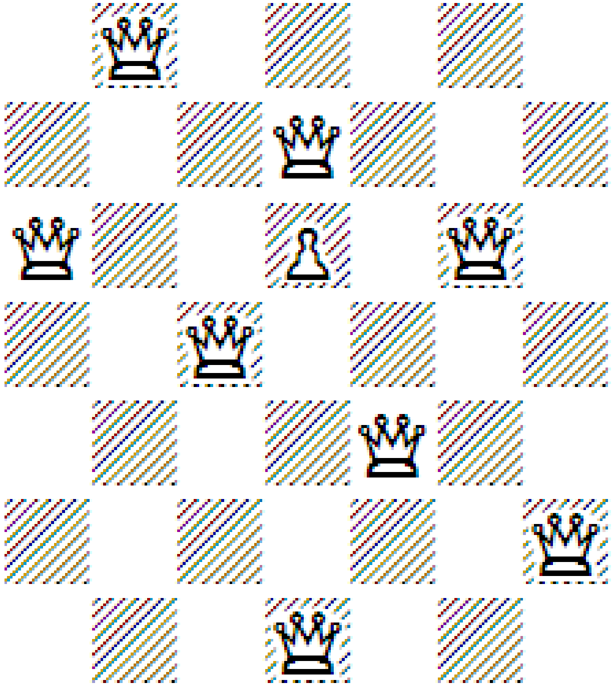


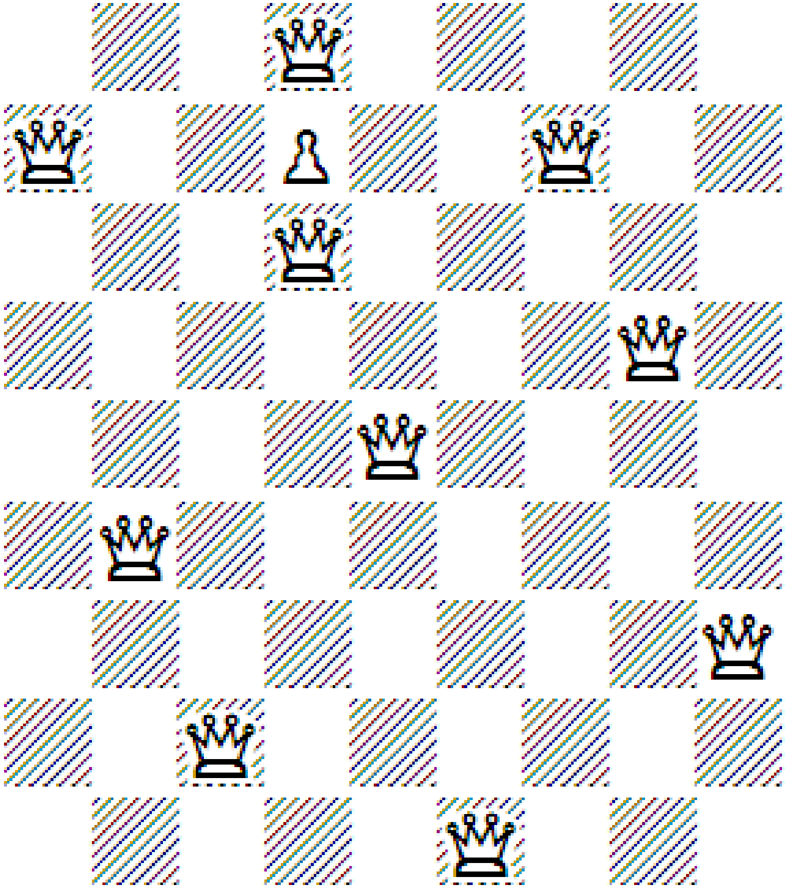
Pattern IV: N ≥15, N ≡ 3 (mod 6)

- Take Construction A solution to (N-3)-Queens.
- Add 3 columns to left, 2 rows on top, 1 row on bottom.
- Pawn: ((N-3)/2, 2)
- Extra Queens ((N-3)/2,-3), (N-2, -2), (-1,-1), (N-3,2)



Final Cases: N = 7 and N = 9





Counting N + 1 Queens Solutions

N	Number of solutions	N	Number of solutions
4	0	11	11,152
5	0	12	65,172
6	16	13	437,848
7	20	14	3,118,664
8	128	15	23,387,448
9	396	16	183,463,680
10	2,288	17	1,474,699,536

Fundamental Solutions

N	Number of solutions	N	Number of solutions
4	0	11	1,403
5	0	12	8,214
6	2	13	54,756
7	3	14	389,833
8	16	15	2,923,757
9	52	16	22,932,960
10	286		

Independence Separation: Adding k Queens

- **Theorem 2**: For each k, for large enough N we have $s_Q(\beta,N+k,N) = k$.
  - Proof is like k=1 proof, but uses more patterns.

How large is "large enough"?

- The proof gives max{87+k,25k} as an upper bound.
- With computer searches, we find
  - N = 7 is large enough for k = 2 (and N = 6 isn't)
  - N = 8 is large enough for k = 3 (and N = 7 isn't)

Counting N + 2 Queens Solutions

N	Number of solutions	N	Number of solutions
5	0	10	1,304
6	0	11	12,452
7	4	12	105,012
8	44		
9	280		

Fundamental Solutions

N	Number of solutions	N	Number of solutions
5	0	10	164
6	0	11	1,572
7	1	12	13,133
8	6		
9	37		

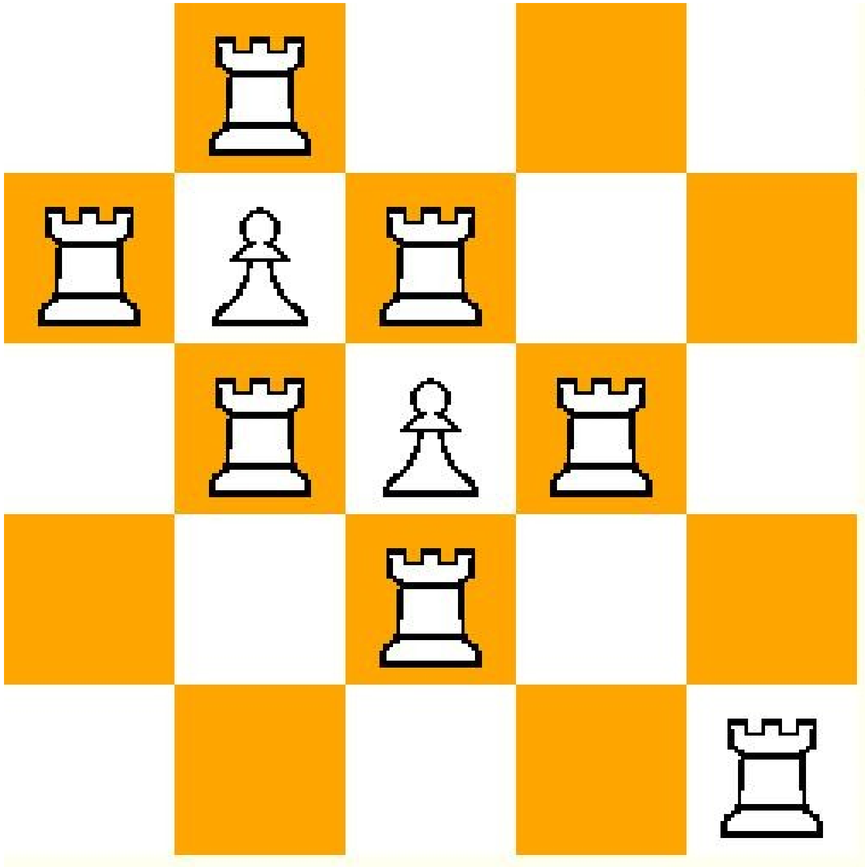
Counting N + 3 Queens Solutions

N	Number of solutions
7	0
8	8
9	44
10	528
11	5,976

Fundamental Solutions

N	Number of solutions
7	0
8	1
9	6
10	66
11	751

Rooks Independence Separation

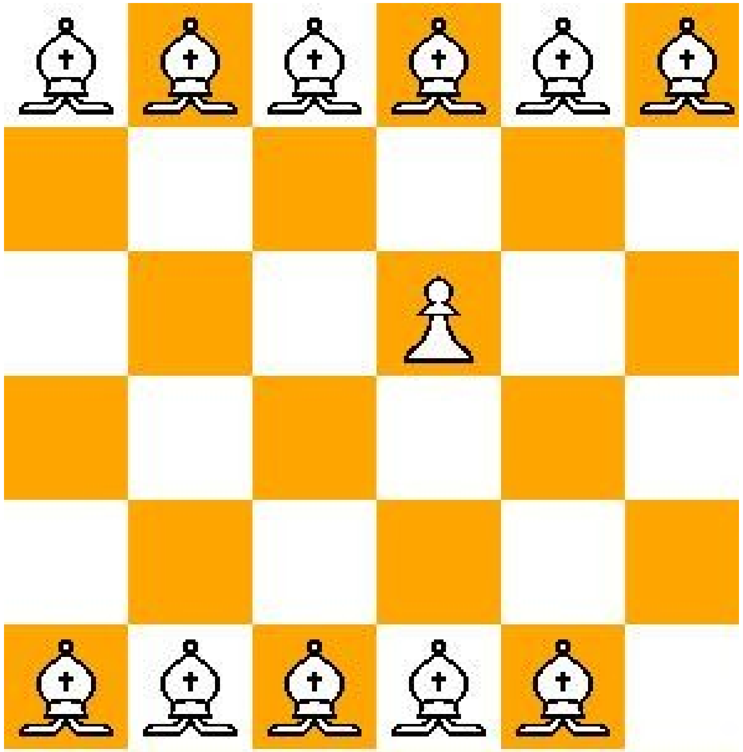


For N ≥ k+2,

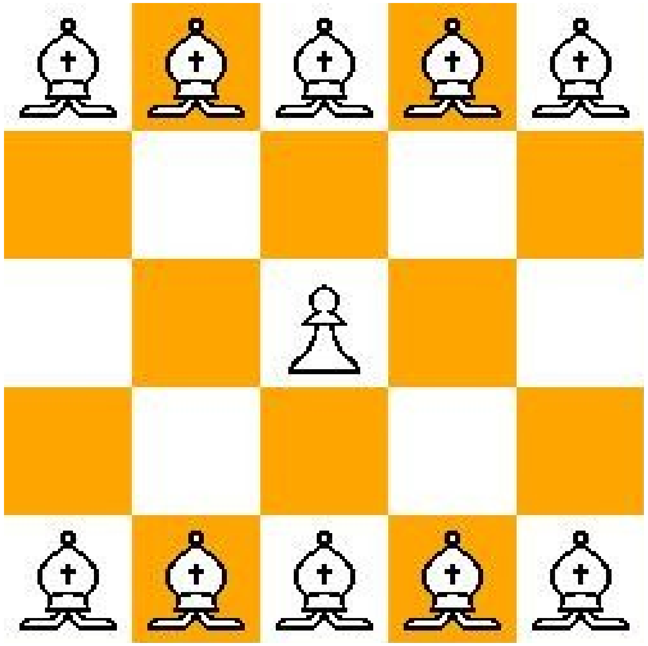
$s_R(\beta, N+k, N) = k$.

Bishops Independence Separation

$s_B(\beta,2n-1,n) = 1$ for $n>2$

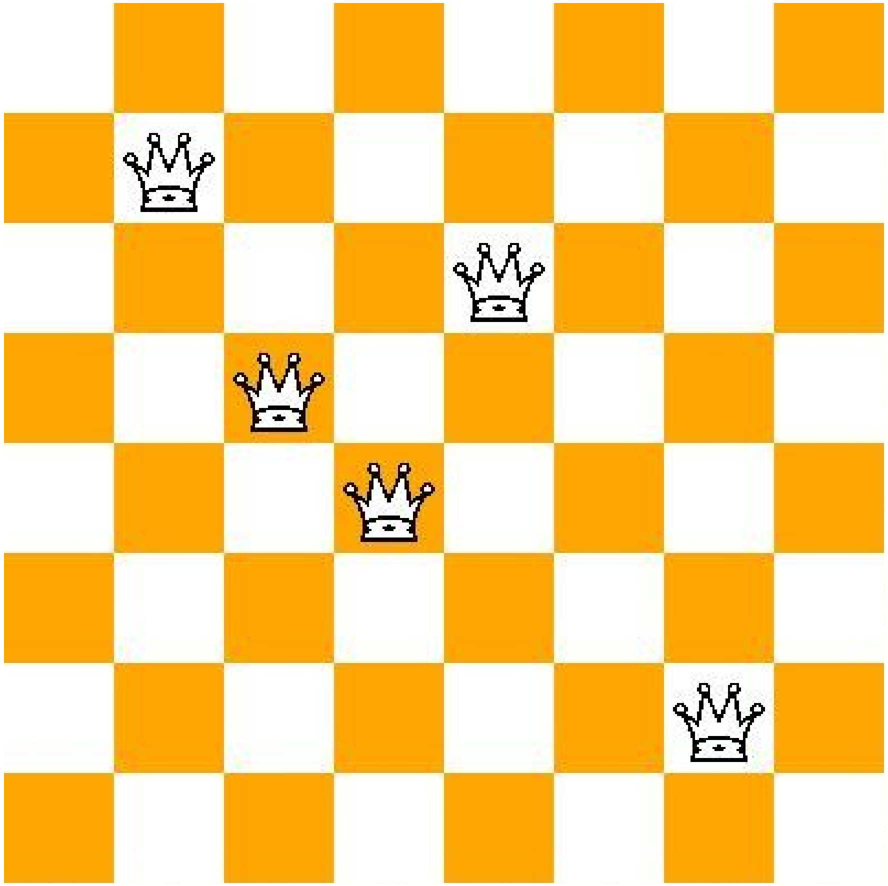


$s_B(\beta,2n,n) = 1$ for $n>2$ odd



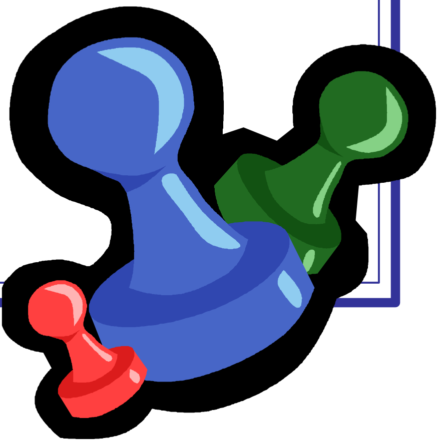
The Five Queens Problem

- Place 5 queens on the chessboard so all squares are either occupied or attacked.
- Introduced in 1862.
- Domination number

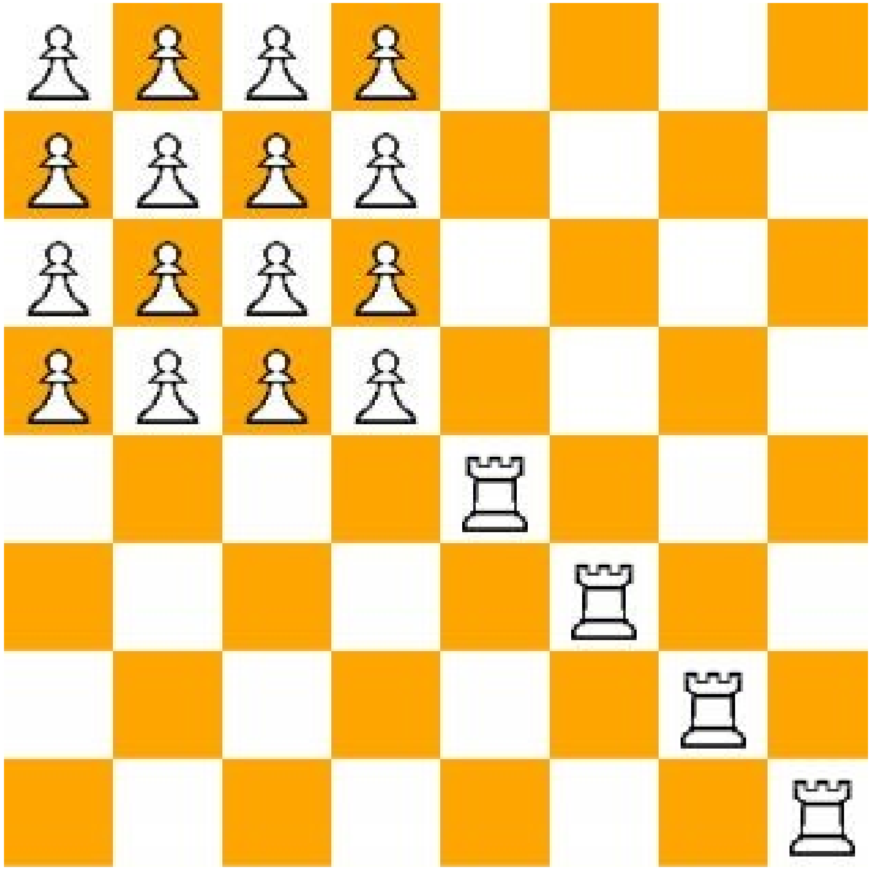


Domination Separation Numbers

- **$s_Q(\gamma,k,n)$** is the minimum number of Pawns we need to place on an n-by-n chessboard so that the Queens graph on the remaining squares has **domination** number k.
- Similar definition for other pieces.



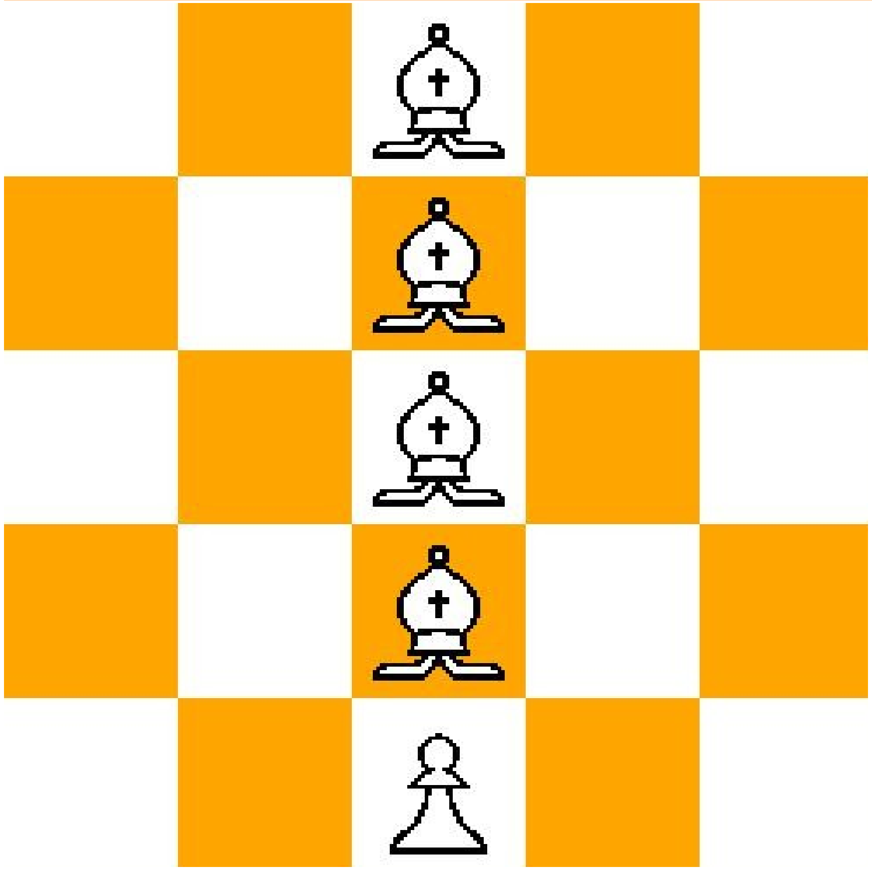
Domination Separation



With enough pawns, we can decrease the domination number.

E.g., for $n \geq k$, $s_R(\gamma, n-k, n) = k^2$

Domination Separation (p. 2)

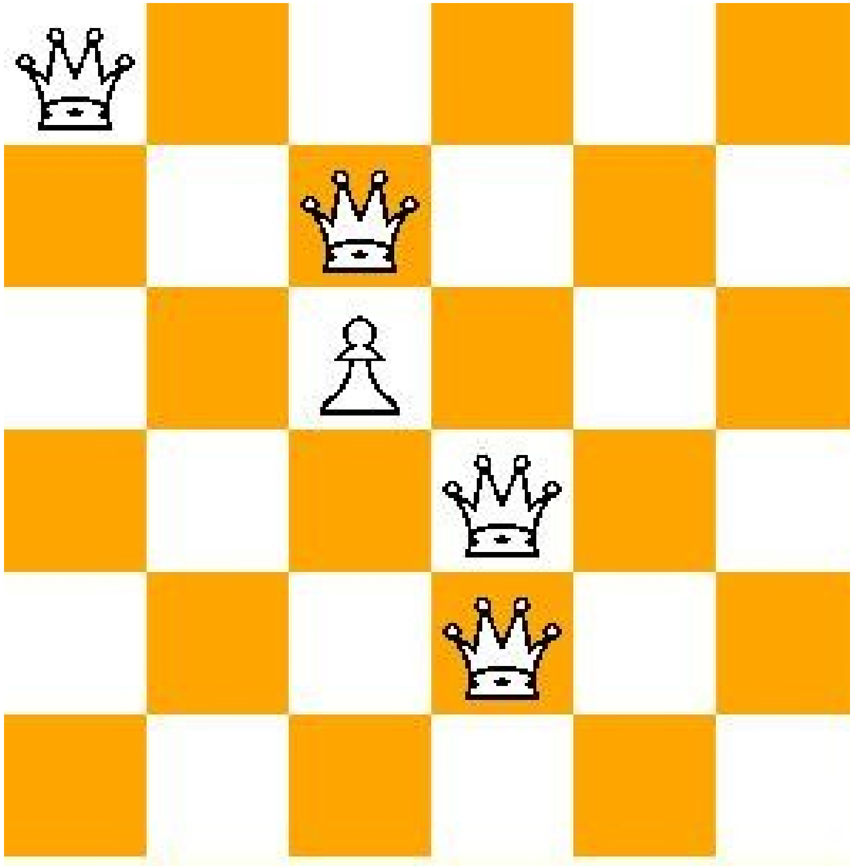


- For $n > 2$ odd, $s_B(\gamma, n\text{-}1, n) = 1$
- For $n \geq 2$ even, $s_B(\gamma, n\text{-}1, n) > 1$

Domination Separation (p.3)

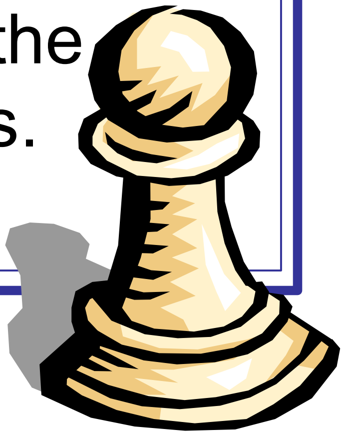
We can sometimes increase the domination number by adding pawns.

$s_Q(\gamma,4,6) = 1$



Open Problems

- Alternate boards (rectangular, toroidal, etc.)
- How many solutions?
- Where can the Pawns go?
  - **Proposition**: If N + k mutually nonattacking Queens and k Pawns are placed on an N-by-N board, then none of the Pawns are in the first or last row or column, nor are any Pawns in the squares diagonally adjacent to the corners.



Open Problems (p. 2)

- Alternate domination parameters.
- Alternate pieces (such as Amazon = Q+N)
- Consider "upper π separation numbers," where we look for the **maximum** number of Pawns needed to get a particular value for π.

References

- Chatham, Fricke, and Skaggs, The Queens Separation Problem, Utilitas Mathematica 69 (2006),129-141.
  - Preprint at http://people.moreheadstate.edu/fs/d.chatham/queenssep.pdf
- The N+k Queens Problem Page
  - http://people.moreheadstate.edu/fs/d.chatham/n+kqueens.html

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- Watkins, John J. (2004). *Across the Board: The Mathematics of Chess Problems*. Princeton: Princeton University Press. ISBN 0-691-11503-6.
- Zhao, Kaiyan (1998), The Combinatorics of Chessboards (Ph. D. Thesis), CUNY.

Your move!

- Any questions?