Reflections on the $n + k$ dragon kings problem

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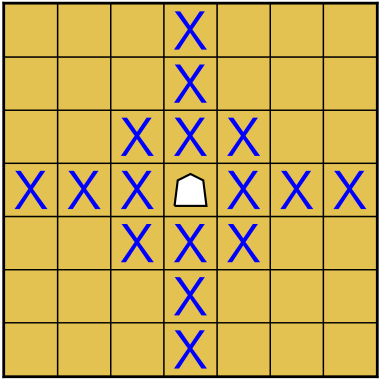
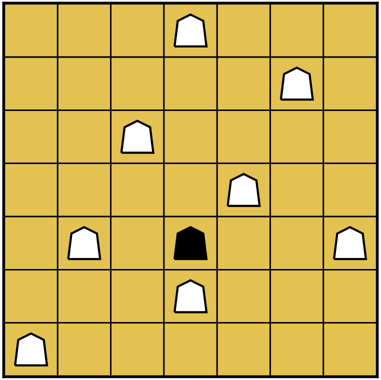
Dragon King: Rook + King



$N + k$ Dragon Kings Problem

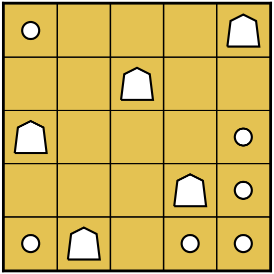
- $n + k$ dragon kings, k pawns on $n \times n$ chessboard
- no two dragon kings attack each other



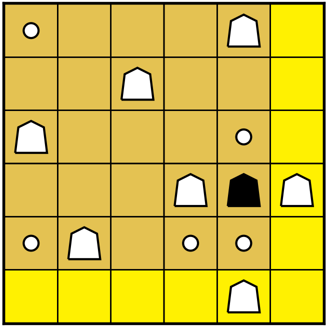
Existence of solutions

Proposition

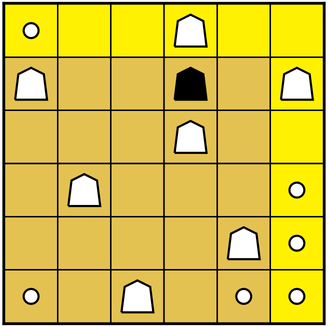
There is a solution when $k \geq 0$ and $n \geq k + 5$.



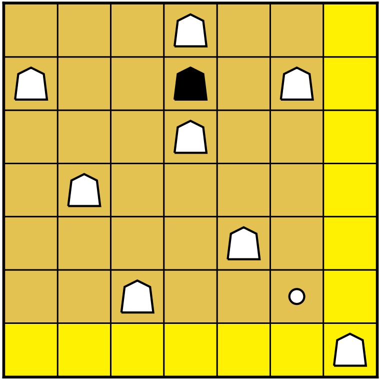
$n = k + 5$



$n = k + 5$: continued

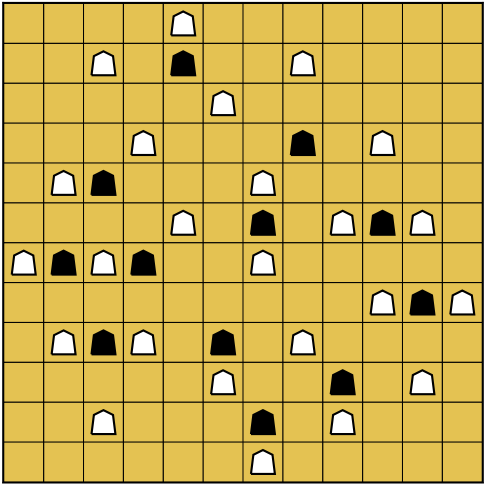


$n \geq k + 5$



Proposition

There is a solution when $k \geq 12$ *and* $n \geq k$.



Number of solutions

For $k = 0$, see OEIS Sequence A002464.

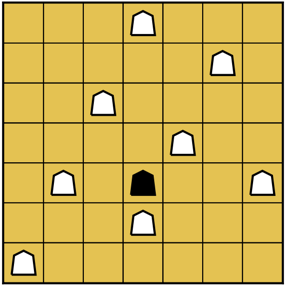
$n \backslash k$	0	1	2	3	4
4	2	0	0	0	0
5	14	0	0	0	0
6	90	32	0	0	0
7	646	762	124	0	0
8	5242	14412	9056	1688	94
9	47622	250326	380776	216678	48374
10	479306	4252504	12538132	16006424	9629406

Number of $n + k$ dragon kings problem solutions
for $4 \leq n \leq 10$ and $0 \leq k \leq 4$

Symmetries

Lemma

Each solution corresponds to an $n \times n$ alternating sign matrix with k -1s in which no two 1's are adjacent.

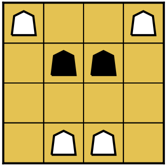


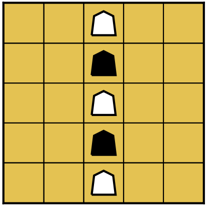
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Symmetries

Proposition

No solution (with $n > 1$) is symmetric with respect to vertical or horizontal reflection.



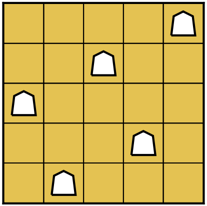


Corollary

We can partition the solution set into 5 symmetry classes.

1. Ordinary

NOT symmetric w.r.t. reflection or nontrivial rotation

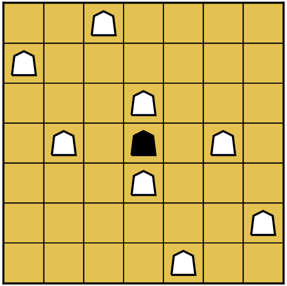


Proposition

These exist for $n \geq k + 5$*.*

2. Centrosymmetric

Symmetric w.r.t. 180° rotation, but not 90° rotation or any reflection

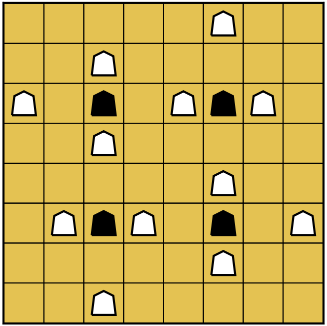


Proposition

If k even, these exist for $n \geq 2k + 6$.
If n and k odd, these exist for $n \geq 2k + 5$.

3. Doubly centrosymmetric

Symmetric w.r.t. 90° rotation, but not any reflection



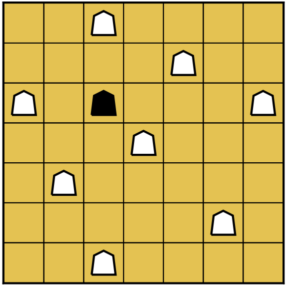
Proposition

A doubly centrosymmetric solution exists if any of the following conditions hold:

1. $n \equiv 0 \pmod 4$, $k \equiv 0 \pmod 4$, *and* $n \geq \max\{2k, 4\}$
2. $n \equiv 1 \pmod 4$, $k \equiv 0 \pmod 4$, *and* $n \geq \max\{2k + 1, 5\}$
3. $n \equiv 3 \pmod 4$, $k \equiv 1 \pmod 4$, *and* $n \geq \max\{2k + 1, 11\}$

4. Monodiagonally symmetric

Symmetric w.r.t. reflection across either main diagonal or antidiagonal, but not both. Not symmetric w.r.t. rotation.

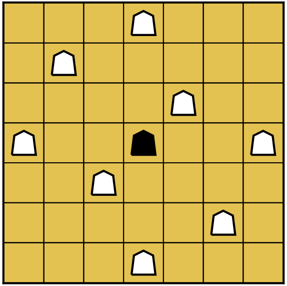


Proposition

These exist for $n \geq 2k + 5$.

5. Bidiagonally symmetric

Symmetric w.r.t. reflection across both the main diagonal and antidiagonal. Also symmetric w.r.t. 180° rotation, but not 90° rotation.



Proposition

A bidiagonally symmetric solution exists if

- k *even and* $n \geq 2k + 6$, *or*
- n *and* k *odd and* $n \geq 2k + 5$

Open problems

Many questions remain.

- How much can we tighten the bounds?
- How many solutions?
- What happens on other types of board (cylinder, torus, etc.)?
- What happens with different restrictions (e.g, not allowing pieces on adjacent squares)?

References

- H. Bodlaender, F. Duniho, Shogi: Japanese chess, 2017. http://www.chessvariants.com/shogi.html
- D. Chatham, Independence and domination on shogiboard graphs, *Recreational Mathematics Magazine*, **8**(2017), 25–37.
- R.D. Chatham, M. Doyle, R.J. Jeffers, W.A. Kosters, R.D. Skaggs, J.A. Ward, Centrosymmetric solutions to chessboard separation problems, *Bulletin of the Institute of Combinatorics and its Applications*, **65**(2012), 6–26.
- MiniZinc:available at http://www.minizinc.org/

Any questions?